Temporal Graph Realization With Bounded Stretch*

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Abstract

A periodic temporal graph, in its simplest form, is a graph in which every edge appears exactly once in the first Δ time steps, and then it reappears recurrently every Δ time steps, where Δ is a given period length. From a network design perspective, a crucial task is to assign the time-labels on the edges in a way that optimizes some criterion. In this paper we introduce a very natural optimality criterion that captures how the temporal distances of all vertex pairs are "stretched", compared to their distances in the underlying static graph. Given a static graph G, the task is to assign to each edge one time-label between 1 and Δ such that, in the resulting periodic temporal graph with period Δ, the duration of the fastest temporal path from any vertex u to any other vertex v is at most α times the distance between u and v in G. Here, the value of α measures how much the shortest paths are allowed to be stretched once we assign the periodic time-labels. Our

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*A preliminary version of this work appeared in the Proceedings of the 50th International Symposium on Mathematical Foundations of Computer Science [40].

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[1]Supported by the EPSRC grant EP/P020372/1.

[2]Supported by the Israel Science Foundation, grant nr. 1470/24, by the European Union's Horizon Europe research and innovation programme under grant agreement 949707, and by the European Research Council, grant nr. 101039913 (PARAPATH).

[3]Supported by the French ANR, project ANR-22-CE48-0001 (TEMPOGRAL).

[4]Supported by the EPSRC grant EP/P02002X/1.

results span three different directions: First, we provide a series of approximation and NP-hardness results. Second, we provide approximation and fixed-parameter algorithms. Among them, we provide a simple algorithm which guarantees an approximation strictly smaller than Δ. Third, we consider a parameterized local search extension of the problem where we are given the temporal labeling of the graph, but we are allowed to change the time-labels of at most k edges.

Keywords: Temporal graph, periodic temporal labeling, fastest temporal path, graph realization, temporal connectivity, stretch

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1. Introduction

Graph realization is a classical problem where one is asked to decide whether a graph with a certain property, such as prescribed distances between vertex pairs or prescribed degrees for vertices, exists [7, 20, 28, 29]. These types of problems have a natural analog in the temporal graph setting[5]. Here we are given a static graph, and the task is to assign time-labels to each edge in order to create a temporal graph with some desired property. Temporal graph realization is an emerging topic in temporal graph algorithmics, and several connectivity-related properties have been considered. Previous work considered the property of temporal connectivity between every vertex pair or subsets thereof [3, 6, 32, 38], the size of so-called reachability sets [19], exact reachability graphs [21], fastest temporal paths[6] of a prescribed (exact) duration between every vertex pair [23, 33], and fastest temporal paths with a prescribed upper bound on the duration for every vertex pair [39, 41]. The latter two problems have been studied in the *periodic setting*, where every edge reappears after some given period length Δ.

We focus on the last among the mentioned problem settings, where we assume that the input consists of a graph and an upper bound for every (ordered) vertex pair that shall be respected by a fastest temporal path connecting the vertices. This problem is naturally motivated in the area of

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[5]A *temporal graph* is a graph where one or more time-labels are assigned to every edge, indicating at which times this edge is active. We give a formal definition in Section 2.

[6]A *temporal path* uses edges with strictly increasing time-labels. A *fastest* temporal path is a temporal path that minimizes the difference between the arrival time and the starting time. We give a formal definition in Section 2.

network design: we are given a transportation infrastructure and are tasked to develop a (periodic) schedule that guarantees fast delivery. In previous work [39], we showed that this problem is NP-hard even if the input graph is a star and the period length Δ is constant. This suggests that a lot of complexity is "hidden" in the prescribed upper bounds. However, it is arguably unnatural to have arbitrary upper bounds in the input that can be completely unrelated to the topology of the input graph. This leads us to the following problem.

We investigate the setting where we wish the durations of all fastest temporal paths to be at most some factor α times the distances of the respective vertex pairs in the (static) input graph. We then say that the realization has a bounded (multiplicative) *stretch*. This is a well-motivated and well-established concept in network design, especially in the context of spanning trees [1, 2, 8, 16, 17] or temporal spanners [5]. It allows the duration of connections between entities of the network to be related to the physical distance of those entities in a natural way. We formally define the problem as follows. The required terminology is formally defined in Section 2.

Stretched Periodic Temporal Graph Realization (SPTGR)

Input: A graph G (static, undirected), $\Delta \in \mathbb{N}$, and a rational number $\alpha \geq 1$.

Question: Let D_G be the distance matrix of G. Can we assign one label in $\{1, \ldots, \Delta\}$ to every edge such that in the resulting Δ-periodic temporal graph, for every vertex pair u, v the duration of a fastest path from u to v is at most $\alpha \cdot D_G(u, v)$?

Our Contribution. SPTGR is a natural and well-motivated special case of the problem that we investigated in previous work [39], where arbitrary upper bounds were given for the duration of a temporal path from any vertex to any other vertex. In this work, we show that SPTGR is NP-hard, and hard to approximate when we aim to minimize the stretch α. Here, in contrast to [39], the duration of any temporal path from u to v is upper bounded by α times the length of the shortest from u to v in the given static graph G. Formally, we prove the following in Section 3.

- For every $0 < \varepsilon < 1$ and every $c > 1$, SPTGR is NP-hard to approximate within a factor of $\Delta^{1-\varepsilon}$ or within a factor of 2^{n^c}.

Note that a stretch of Δ can trivially be achieved by assigning the same label to each edge. To complement these results, we show that we can achieve a strictly better approximation ratio than Δ, especially on graphs with small radius or diameter. Formally, we prove the following in Section 4.

- We can compute in polynomial time a solution for a SPTGR instance with stretch at most $\Delta - \frac{\Delta-1}{\min(\text{rad}+1,\text{diam})}$, where "diam" and "rad" denote the diameter and the radius of G, respectively.

Next, we focus on the decision version of the problem. We show that SPTGR is NP-hard even if Δ and α are small constants and in cases where the input graphs have a constant diameter. Formally, we prove the following in Section 5.

- SPTGR is NP-hard if $\Delta = 3$, $\alpha = 1$, and the input graph has diameter 2. This answers an open question by Erlebach et al. [23].
- SPTGR is NP-hard for each $\Delta \geq 3$, even if the diameter is in $\mathcal{O}(\Delta)$.

Recall that we can assume that α is at most Δ. We also show that SPTGR is NP-hard for each constant value of $\alpha \geq 1$. To complement these hardness results, we give the following algorithmic results in Section 6. We use standard terminology from parameterized complexity theory [11].

- SPTGR is fixed-parameter tractable when parameterized by the neighborhood diversity of the input graph and Δ.
- SPTGR is fixed-parameter tractable when parameterized by the treewidth of the input graph, the diameter of the input graph, and Δ.

These results are obtained by using extensions of monadic second order logic (MSO) [9, 10, 34] to formulate SPTGR, and then applying extensions of Courcelle's theorem [9, 10, 34].

Finally, we develop a local search algorithm for our problem. This algorithm, given a labeling, checks whether the stretch can be improved by changing at most a given number k of labels. We call k the search radius. We show the following in Section 7.

- Given a periodic labeling for a graph G and some constant k, we can compute the best possible stretch that can be obtained by changing at most k labels in polynomial time; that is, we present an algorithm that is in XP with respect to the parameter k.

- We show that the above-described local search problem is W[2]-hard when parameterized by the search radius k.

Related Work. Since the 1960s, graph realization problems have been studied in many different settings. A complete literature review is beyond the scope of this work. While in the static setting, many different properties for realization have been studied, in particular, the realization of degree sequences, the previous work in the temporal setting considers mostly connectivity-related properties. We review the relevant work in the introduction.

The local search variant of our problem can also be seen as a temporal graph modification problem: we may modify up to k labels to manipulate the graph's connectivity properties. Many problems in this setting have been studied, where the modifications are typically edge removal, edge delay [14, 18, 37, 42], or vertex removal [25, 31, 46]. Finally, *periodic* temporal graphs also have been investigated in other problem settings, most prominently in the context of cop and robber games [4, 12, 13, 22, 44, 45].

2. Preliminaries

An undirected graph $G = (V, E)$ consists of a set V of vertices and a set $E \subseteq \binom{V}{2}$ of edges. We denote by $V(G)$ and $E(G)$ the vertex and edge set of G, respectively. Whenever no confusion arises, we denote $V(G)$ and $E(G)$ by just V and E, respectively. We use standard concepts and terminology from graph theory like *diameter* and *radius* [15]. The *eccentricity* of a vertex v is the greatest distance between v and any other vertex [15].

Let $G = (V, E)$ and $\Delta \in \mathbb{N}$, and let $\lambda : E \to \{1, \dots, \Delta\}$ be an edge-labeling function that assigns to every edge of G exactly one of the labels from $\{1, \dots, \Delta\}$. Then we denote by (G, λ, Δ) the Δ-*periodic temporal graph* (G, L), where for every edge $e \in E$ we have $L(e) = \{\lambda(e) + i\Delta \mid i \in \mathbb{Z}, i \geq 0\}$. In this case, we call λ a Δ-*periodic labeling* of G. When it is clear from the context, we drop Δ and denote the (Δ-periodic) temporal graph by (G, λ). We assume that Δ is encoded in binary in instances of SPTGR. Hence, the size of an instance is linear in n, m, $\log \Delta$, and the encoding length of α.

In the following definitions of temporal walks and paths, we assume that they need to be *strict*, that is, labels have to be strictly increasing, and we always use the next-possible appearance of an edge in the Δ-periodic labeling and do not unnecessarily wait at any vertex along the walk or path.

Formally, a *temporal* (s,z)*-walk* (or *temporal walk*) of length k from vertex $s = v_0$ to vertex $z = v_k$ in a Δ-periodic temporal graph (G, L) is a sequence $P = ((v_{i-1}, v_i, t_i))_{i=1}^{k}$ of triples that we call *transitions*, such that for all $i \in [k]$ we have that $t_i \in L(\{v_{i-1}, v_i\})$ and for all $i \in [k-1]$ we have that $t_i < t_{i+1} \leq t_i + \Delta$. Note that this implies that each edge is used at its earliest possible appearance. Moreover, we call P a *temporal* (s,z)*-path* (or *temporal path*) of length k if $v_i \neq v_j$ for all $i, j \in \{0, \ldots, k\}$ with $i \neq j$. Given a temporal path $P = ((v_{i-1}, v_i, t_i))_{i=1}^{k}$, we denote the set of vertices of P by $V(P) = \{v_0, v_1, \ldots, v_k\}$. A temporal (s,z)-path $P = ((v_{i-1}, v_i, t_i))_{i=1}^{k}$ is *fastest* if for all temporal (s,z)-path $P' = \left((v'_{i-1}, v'_i, t'_i)\right)_{i=1}^{k'}$ we have that $t_k - t_0 \leq t'_{k'} - t'_0$. We say that the *duration* of P is $d(P) = t_k - t_0 + 1$. Note that adding one to the difference between arrival and starting time models that the edges have transition time one.

Definition 1 (Stretch)**.** *Let $\mathcal{G} := (G, \lambda)$ be a Δ-periodic temporal graph where G is a connected graph. The* stretch *of $\mathcal{G}$ denoted by $\alpha_{\mathcal{G}}$ is value obtained from dividing the duration of the fastest path (*$\mathrm{dur}(u,v)$*) from u to v in $\mathcal{G}$ by the distance (*$\mathrm{dist}(u,v)$*) between u to v in G, maximized over all vertex pairs (u,v). That is, $\alpha_{\mathcal{G}} := \max_{(u,v)\in V\times V, u\neq v} \frac{\mathrm{dur}(u,v)}{\mathrm{dist}(u,v)}$.*

We may omit the subscript if the temporal graph is clear form the context.

Let P be a temporal path on the edges $e_1, \ldots, e_k$ in a Δ-periodic labeling λ of G. For every $i = 1, \ldots, k-1$ let $\lambda(e_i) \in \{1, \ldots, \Delta\}$ be the label assigned to edge e_i, and let v_i be the common vertex of the edges e_i and e_{i+1}. Note that, in the context of *fastest* temporal paths, once a fastest temporal path P starts, each edge of P is visited with the earliest possible appearance. We define the *waiting time* $\mathrm{wait}(v_i)$ of P on vertex v_i to be $\lambda(e_{i+1}) - \lambda(e_i)$ (if $\lambda(e_{i+1}) > \lambda(e_i)$), or $\Delta + \lambda(e_{i+1}) - \lambda(e_i)$ (if $\lambda(e_{i+1}) < \lambda(e_i)$), or Δ (if $\lambda(e_{i+1}) = \lambda(e_i)$). Then, the duration of P is $\sum_{i=1}^{k} \mathrm{wait}(v_i) + 1$.

The next observation follows easily from the fact that the worst-case for the stretch for (u,v) is realized when all edges of the graph have the same time-label.

Observation 2. *Let λ be a Δ-labeling for a graph G. Then for any two vertices u and v, the stretch for (u,v) under λ is at most $\Delta - \frac{\Delta-1}{\mathrm{dist}(u,v)}$.*

Finally, we give a brief argument that we can use polynomially many calls to a decision oracle for SPTGR to find the optimal stretch for a given input graph. Note that naïvely trying out all possible values for α does not yield polynomial time.

Lemma 3. *Given a graph G and some Δ, one can compute the smallest α such that (G, Δ, α) is a yes-instance of* SPTGR *with $\mathcal{O}(\text{diam}(G) \cdot \log(\text{diam}(G) \cdot \Delta))$ calls to a decision oracle for* SPTGR*.*

Proof. Assume we are given a graph G and some Δ. Let $\alpha^\star$ denote the smallest (rational) number such that $(G, \Delta, \alpha^\star)$ is a yes-instance of SPTGR. We know that $1 \leq \alpha^\star \leq \Delta$. We will argue that there are at most $\text{diam}(G) \cdot \Delta$ possible values for $\alpha^\star$.

Let D denote the distance matrix of G. Note that the entries in D are between (and including) 1 and $\text{diam}(G)$. Each value for $\alpha^\star$ yields a matrix of upper bounds that the fastest temporal paths in the periodic temporal graph (that we try to find) have to obey. If a value in the distance matrix is d, then the corresponding value in the upper bounds matrix is $\lfloor \alpha^\star \cdot d \rfloor$. Hence, for each value d, there are at most $d \cdot \Delta$ different possible upper bounds. As we argued earlier, there are $\text{diam}(G)$ possible values for d. This yields $\text{diam}(G)^2 \cdot \Delta$ possible values for $\alpha^\star$ such that $\lfloor \alpha^\star \cdot d \rfloor = \alpha^\star \cdot d$ for some value d between 1 and $\text{diam}(G)$.

Now, in order to perform binary search on these values, we fix some d and find the smallest $\alpha^\star$ such that $\lfloor \alpha^\star \cdot d \rfloor = \alpha^\star \cdot d$ and $(G, \Delta, \alpha^\star)$ is a yes-instance of SPTGR. Note that we can analytically compute each relevant value for $\alpha^\star$ in constant time, and hence need $\mathcal{O}(\log(\text{diam}(G) \cdot \Delta))$ decision oracle calls. By doing this for each possible value d, we can find the overall smallest $\alpha^\star$ with the claimed number of $\mathcal{O}(\text{diam}(G) \cdot \log(\text{diam}(G) \cdot \Delta))$ calls to a decision oracle for SPTGR. □

3. Approximation Hardness

In this section, we show that SPTGR is NP-hard to approximate. In particular, we rule out constant factor approximations and even approximation algorithms with approximation factors that are sublinear in Δ or single exponential in n. Formally, we show the following.

Theorem 4. *Assume that P $\neq$ NP. Then, for all constants $0 < \varepsilon < 1$ and $c \geq 1$:*

- *there is no polynomial-time $\Delta^{1-\varepsilon}$-approximation algorithm for* SPTGR*;*

- *there is no polynomial-time 2^{n^c}-approximation algorithm for* SPTGR.

Proof. We present a straightforward gap-introducing reduction from Gossiping which is known to be NP-hard [26], and thus there is no polynomial-time algorithm that distinguishes between yes-instances and no-instances of Gossiping (assuming that P $\neq$ NP).

Gossiping	
Input:	A graph $G = (V, E)$.
Question:	Is there a labeling $\lambda\colon E \to \mathbb{N}$, such that for each pair (u, v) of vertices of V, there is a temporal path in the temporal graph (G, λ)?

Given an instance G of Gossiping, we produce an instance of SPTGR as follows: We use the same graph G and set $\Delta > \frac{n}{2} \cdot \binom{n}{2}^2$. The concrete value of Δ depends on the value of ε and c respectively and will be specified later.

Intuitively, if G is a yes-instance of Gossiping, then there exists a labeling where it is not necessary to use any label from the second Δ-period and hence the stretch is independent from Δ. Whereas if G is a no-instance of the gossiping problem, then there exists no such labeling. Then, informally speaking, there must be a fastest temporal path between some vertex pair that starts at the first period and ends in the last period, which implies that the duration of that path depends on Δ. This then implies that the optimal stretch depends on Δ, too. By setting Δ to be at least n^5, we then create a *gap* between the worst possible stretch of any yes-instance of Gossiping and the best possible stretch of any no-instance of Gossiping.

Formally, assume that G is a yes-instance of Gossiping and let λ be a labeling such that the non-periodic temporal graph (G, λ) is temporally connected. We can assume without loss of generality that the largest label in λ is at most $\binom{n}{2}$.[7] Now we use this labeling for our SPTGR instance. A very naïve estimation yields that the stretch is at most $\frac{1}{2} \cdot \binom{n}{2}$: Consider a

[7]As G has at most $\binom{n}{2}$ edges, these can have at most $\binom{n}{2}$ different labels in λ. If the largest label in λ is greater than $\binom{n}{2}$, then we can define a different labeling λ' such that (i) the largest label in λ' is at most $\binom{n}{2}$ and (ii) for every two edges e_1, e_2, the relative order of the labels $\lambda'(e_1)$ and $\lambda'(e_2)$ is the same as the relative order of the labels $\lambda(e_1)$ and $\lambda(e_2)$. Then (G, λ') is temporally connected as (G, λ) is also temporally connected.

vertex pair u, v of distance 2 in G. Then the temporal path from u to v in (G, λ) has duration at most $\binom{n}{2}$.

Now assume that G is a no-instance of GOSSIPING and consider the instance (G, Δ) of SPTGR. Let λ be a Δ-periodic labeling for G that minimizes the stretch. Note that we can obtain an equivalent labeling by adding a constant (modulo Δ) to every label. Hence, assume that $\delta = \Delta - \max_{e \in E(G)} \lambda(e)$ is maximized. Then we have that $\delta \geq \frac{\Delta}{\binom{n}{2}}$. Since G is a no-instance of the gossiping problem, there is a vertex pair u, v in G such that the temporal path from u to v in the Δ-periodic temporal graph (G, λ) must use labels from different periods and hence has duration at least δ and hence a stretch of at least $\frac{\delta}{n}$.

Let $\alpha^\star$ denote the optimal stretch of (G, Δ). Summarizing, we have the following.

- If G is a yes-instance of GOSSIPING, then we have $\alpha^\star \leq \frac{1}{2} \cdot \binom{n}{2} =: \alpha_{\text{yes}}$.
- If G is a no-instance of GOSSIPING, then we have $\alpha^\star \geq \frac{\Delta}{n \cdot \binom{n}{2}} =: \alpha_{\text{no}}$.

Now note that $\frac{\alpha_{\text{no}}}{\alpha_{\text{yes}}} = 2 \cdot \frac{\Delta}{n \cdot \binom{n}{2}^2} > \frac{\Delta}{n^5}$. Therefore, if we set $\Delta > n^5$, we get that $\frac{\alpha_{\text{no}}}{\alpha_{\text{yes}}} > 1$.

For the first statement of the theorem, let $0 < \varepsilon < 1$ and let $\varepsilon' = \frac{5(1-\varepsilon)}{\varepsilon}$. Note that $1 - \varepsilon = \frac{\varepsilon'}{5+\varepsilon'}$. Now we set $\Delta = n^{5+\varepsilon'}$. Then we have that $\Delta^{1-\varepsilon} = \Delta^{\frac{\varepsilon'}{5+\varepsilon'}} = n^{(5+\varepsilon') \cdot \frac{\varepsilon'}{5+\varepsilon'}} = n^{\varepsilon'}$. It follows that $\frac{\alpha_{\text{no}}}{\alpha_{\text{yes}}} > \frac{\Delta}{n^5} = n^{\varepsilon'} = \Delta^{1-\varepsilon}$. Now, assume that there exists a polynomial-time approximation algorithm for SPTGR with ratio at most $\Delta^{1-\varepsilon}$. Then, since $\frac{\alpha_{\text{no}}}{\alpha_{\text{yes}}} > \Delta^{1-\varepsilon}$, we can use this polynomial-time algorithm to distinguish between yes-instances and no-instances of GOSSIPING, which is a contradiction assuming that P $\neq$ NP.

For the second statement of the theorem, let $c \geq 1$ and set $\Delta = n^5 \cdot 2^{n^c}$. Note that the encoding of Δ is polynomial in n, as $\log \Delta = n^c + 5 \log n$. Then we have that $\frac{\alpha_{\text{no}}}{\alpha_{\text{yes}}} > \frac{\Delta}{n^5} = 2^{n^c}$. Similarly to the above, assume that there exists a polynomial-time approximation algorithm for SPTGR with ratio at most 2^{n^c}. Then, since $\frac{\alpha_{\text{no}}}{\alpha_{\text{yes}}} > 2^{n^c}$, we can use this polynomial-time algorithm to distinguish between yes-instances and no-instances of GOSSIPING, which is again a contradiction assuming that P $\neq$ NP. □

4. Approximation Algorithm

In this section, we give an approximation algorithm for SPTGR that runs in polynomial time and achieves an approximation ratio of $\Delta - \frac{\Delta-1}{\min(\text{rad}+1, \text{diam})}$, where *diam* and *rad* denote the diameter and the radius of the input graph G, respectively. Formally, we show the following.

Theorem 5. *A solution for* SPTGR *with stretch at most* $\Delta - \frac{\Delta-1}{\min(\text{rad}+1, \text{diam})}$ *can be computed in polynomial time.*

Note that this is strictly better than the "trivial" stretch obtained by Observation 2 if the radius and diameter differ. To show Theorem 5, we give the following algorithm, which we will refer to as *the radius algorithm.* Assume we are given an instance (G, Δ) of (the optimization version of) SPTGR. We perform the Algorithm 1.

Algorithm 1 The radius algorithm.

Input: A graph G and a period Δ.
Output: A labeling $\lambda\colon E(G) \to \{1, \dots, \Delta\}$.

```
Initialize λ(e) := 1 for each edge e ∈ E(G)
v_x ← vertex of V(G) with smallest eccentricity
rad ← the radius of G
for all i ∈ [1, rad] do
    for all e ∈ E(N^{i-1}(v_x), N^i(v_x)) do      ▷ For each k ∈ ℕ, N^k(v_x)
        denotes the set of all vertices of distance exactly k with v_x.
        if i is odd then
            λ(e) := ⌈Δ/2⌉
        else
            λ(e) := Δ
Return λ
```

For an illustration see Figure 1. We show that this algorithm achieves the following stretch.

Lemma 6. *Let* $(G = (V, E), \Delta)$ *be an instance of* SPTGR. *Then the radius algorithm computes a labeling with stretch at most* $\Delta - \frac{\Delta-1}{\min(\text{rad}+1, \text{diam})}$.

Proof. Let v_x be an arbitrary vertex of eccentricity equal to rad and let λ be the Δ-labeling produced by the radius algorithm for root v_x. We give for

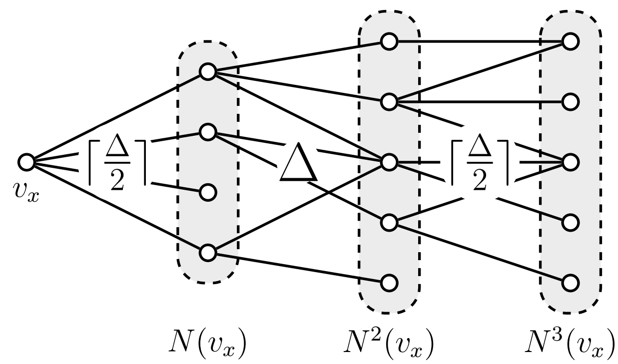

Figure 1: Example of a graph G with radius 3, where $v_x \in V(G)$ is a vertex of eccentricity equal to the radius. The gray areas depict the distance 1-3 neighborhoods of v_x. The labels given by the radius algorithm are illustrated. Edges between vertices in the same neighborhood are not depicted and are given arbitrary labels by the algorithm.

each distance $\ell \in [2, \text{diam}]$ an upper bound α_ℓ for the stretch of distance-ℓ vertex pairs.

For $\ell \leq \text{rad}+1$, let (u, v) be a pair of vertices of distance exactly ℓ in G. Due to Observation 2, $\alpha_\ell := \Delta - \frac{\Delta-1}{\ell}$ is an upper bound for the stretch of distance-ℓ vertex pairs.

For $\ell \geq \text{rad}+2$, let (u, v) be a pair of vertices of distance exactly ℓ in G. Consider the journey J that starts in u, goes over any shortest path P_u to v_x and then goes over any shortest path P_v to v. The edges of P_u (P_v) are alternatively labeled with Δ and $\lceil \frac{\Delta}{2} \rceil$ by the algorithm. Moreover, each of the two paths P_u and P_v has a length of at most rad. Hence, in the worst case, J traverses $2 \cdot \text{rad}$ edges. This implies that the duration of J is at most $\text{rad} \cdot \Delta + 1$, since J only waits a full period at vertex v_x and otherwise alternates between waiting $\lfloor \frac{\Delta}{2} \rfloor$ and $\lceil \frac{\Delta}{2} \rceil$. Note that this also implies that there is a path of at most that duration from u to v. Consequently, $\alpha_\ell := \frac{\text{rad} \cdot \Delta+1}{\ell} = \Delta + \frac{(\text{rad} - \ell)\cdot\Delta+1}{\ell} = \Delta - \frac{\Delta-1}{\ell} - \frac{(\ell - \text{rad} - 1)\cdot\Delta}{\ell}$ is an upper bound for the stretch of distance-ℓ vertex pairs.

Note that the stretch achieved by λ is thus at most $\max\{\alpha_\ell \mid 2 \leq \ell \leq \text{diam}\}$, if $\text{diam} \geq 2$. (For $\text{diam} = 1$, the stretch is always 1.) We show that the maximum is achieved for $\ell = \text{rad}+1$ if $\text{rad} < \text{diam}$, and for $\ell = \text{rad}$ if $\text{rad} = \text{diam}$.

To this end, first note that $\alpha_{\text{rad}} = \Delta - \frac{\Delta-1}{\text{rad}} \geq \max\{\alpha_\ell = \Delta - \frac{\Delta-1}{\text{rad}} \mid \ell \in [2, \text{rad}]\}$. This proves the statement if $\text{rad} = \text{diam}$. Otherwise, assume that $\text{rad} < \text{diam}$. If $\text{diam} = \text{rad}+1$, then the statement also holds, since $\alpha_{\text{rad}+1} = \Delta - \frac{\Delta-1}{\text{rad}+1} - \frac{(\text{rad}+1-\text{rad}-1)\cdot\Delta}{\text{rad}+1} = \Delta - \frac{\Delta-1}{\text{rad}+1} > \Delta - \frac{\Delta-1}{\text{rad}} = \alpha_{\text{rad}}$. Thus, consider $\text{diam} \geq \text{rad}+2$. Note that this implies that $\text{rad} \geq 2$, since $\text{rad} \geq \frac{\text{diam}}{2}$. We show that $\alpha_{\text{rad}} \geq \alpha_{\text{rad}+2} \geq \max\{\alpha_\ell \mid \text{rad}+2 \leq \ell \leq \text{diam}\}$.

First, we show $\alpha_{\text{rad}} \geq \alpha_{\text{rad}+2}$.

$$\begin{aligned}
\alpha_{\text{rad}} = \Delta - \frac{\Delta-1}{\text{rad}} &\geq \Delta - \frac{\Delta-1}{\text{rad}+2} - \frac{(\text{rad}+2-\text{rad}-1)\cdot\Delta}{\text{rad}+2} = \alpha_{\text{rad}+2} \\
\Leftrightarrow \frac{\Delta-1}{\text{rad}+2} + \frac{\Delta}{\text{rad}+2} &\geq \frac{\Delta-1}{\text{rad}} \\
\Leftrightarrow \text{rad}(\Delta-1) + \text{rad}\,\Delta &\geq (\text{rad}+2)(\Delta-1) \\
\Leftrightarrow \text{rad}\,\Delta &\geq 2(\Delta-1) \\
\Leftrightarrow 2 + (\text{rad}-2)\Delta &\geq 0
\end{aligned}$$

This holds true, since $\text{rad} \geq 2$.

It remains to show that $\alpha_{\text{rad}+2} \geq \max\{\alpha_\ell \mid \text{rad}+2 \leq \ell \leq \text{diam}\}$. To this end, let $\ell \in [\text{rad}+2, \text{diam}-1]$. We show that $\alpha_\ell \geq \alpha_{\ell+1}$. Recall that $\alpha_\ell = \Delta + \frac{(\text{rad}-\ell)\cdot\Delta+1}{\ell}$.

$$\begin{aligned}
\alpha_\ell = \Delta + \frac{(\text{rad}-\ell)\cdot\Delta+1}{\ell} &\geq \Delta + \frac{(\text{rad}-\ell-1)\cdot\Delta+1}{\ell+1} = \alpha_{\ell+1} \\
\Leftrightarrow \frac{(\text{rad}-\ell)\cdot\Delta+1}{\ell} &\geq \frac{(\text{rad}-\ell-1)\cdot\Delta+1}{\ell+1} \\
\Leftrightarrow (\ell+1)(\text{rad}-\ell)\cdot\Delta + \ell + 1 &\geq \ell(\text{rad}-\ell-1)\cdot\Delta + \ell \\
\Leftrightarrow (\ell)(\text{rad}-\ell)\cdot\Delta + (\text{rad}-\ell)\cdot\Delta + \ell + 1 &\geq \ell(\text{rad}-\ell)\cdot\Delta - \ell\Delta + \ell \\
\Leftrightarrow (\text{rad}-\ell)\cdot\Delta + 1 &\geq -\ell\Delta \Leftrightarrow \text{rad}\cdot\Delta + 1 \geq 0
\end{aligned}$$

Consequently, for $\text{diam} \geq \text{rad}+2$, $\alpha_{\text{rad}+1} = \Delta - \frac{\Delta-1}{\text{rad}+1} \geq \max\{\alpha_\ell \mid \ell \in [2, \text{diam}]\}$.

This thus proves that the stretch achieved by λ is at most $\Delta - \frac{\Delta-1}{\min(\text{rad}+1,\text{diam})}$. □

Theorem 5 follows directly from Lemma 6 and the straightforward observation that the radius algorithm runs in polynomial time. However, we can show that in several restricted cases, the algorithm is guaranteed to achieve a better stretch.

Lemma 7. *Let* $(G = (V, E), \Delta)$ *be an instance of* SPTGR. *If* $2 \leq \text{rad} < \text{diam}$ *and there is a root* v_x *such that for each distance-*$(\text{rad} + 1)$ *vertex* *pair* (u, v)*,* $\text{dist}(v_x, u) + \text{dist}(v_x, v) < 2\,\text{rad}$*, then the radius algorithm (for root* v_x*) computes a labeling with stretch at most* $\Delta - \frac{\Delta-1}{\text{rad}}$*.*

Proof. Similar to the proof of Lemma 6, we give for each distance $\ell \in [2, \text{diam}]$ an upper bound α_ℓ for the stretch of distance-ℓ vertex pairs.

Due to the proof of Lemma 6, $\alpha_\ell := \Delta - \frac{\Delta-1}{\ell}$ is an upper bound for the stretch of distance-ℓ vertex pairs with $\ell \leq \text{rad} + 1$, and $\alpha_\ell := \Delta - \frac{\Delta-1}{\ell} - \frac{(\ell-\text{rad}-1)\cdot\Delta}{\ell}$ is an upper bound for the stretch of distance-ℓ vertex pairs with $\ell \geq \text{rad} + 2$. Moreover, the maximum over all these stretches was achieved for $\ell = \text{rad}+1$. We now show that $\beta_{\text{rad}+1} := \Delta - \frac{\Delta-1}{\text{rad}+1} - \frac{\lfloor \frac{\Delta}{2} \rfloor}{\text{rad}+1}$ is an upper bound for the stretch of distance-$(\text{rad}+1)$ vertex pairs, as we assume that for each distance-$(\text{rad}+1)$ vertex pair (u, v), $\text{dist}(v_x, u) + \text{dist}(v_x, v) < 2\,\text{rad}$. Additionally, we will show that $\alpha_{\text{rad}} \geq \max\{\alpha_\ell \mid 2 \leq \ell \leq \text{diam}, \ell \neq \text{rad}+1\} \cup \{\beta_{\text{rad}+1}\}$. This then implies that the total stretch of λ is at most $\alpha_{\text{rad}} = \Delta - \frac{\Delta-1}{\text{rad}}$.

Let (u, v) be a pair of vertices of distance exactly ℓ in G. Consider the journey J that starts in u, goes over any shortest path P_u to v_x, and then goes over any shortest path P_v to v. The edges of P_u (P_v) are alternatively labeled with Δ and $\lceil \frac{\Delta}{2} \rceil$ by the algorithm. Moreover, by our assumption, J traverses at most $2\text{rad}-1$ edges. This implies that the duration of J is at most $\text{rad}\cdot\Delta - \lfloor \frac{\Delta}{2} \rfloor + 1$, since J only waits a full period at vertex v_x and otherwise alternates between waiting $\lfloor \frac{\Delta}{2} \rfloor$ and $\lceil \frac{\Delta}{2} \rceil$. Note that this also implies that there is a path of at most that duration from u to v. Consequently $\frac{\text{rad}\cdot\Delta - \lfloor \frac{\Delta}{2} \rfloor + 1}{\text{rad}+1} = \Delta - \frac{\Delta-1}{\text{rad}+1} - \frac{\lfloor \frac{\Delta}{2} \rfloor}{\text{rad}+1} = \beta_{\text{rad}+1}$ is an upper bound for the stretch of distance-$(\text{rad}+1)$ vertex pairs.

It remains to show that $\alpha_{\text{rad}} \geq \max\{\alpha_\ell \mid 2 \leq \ell \leq \text{diam}, \ell \neq \text{rad}+1\} \cup \{\beta_{\text{rad}+1}\}$.

To this end, first note that $\alpha_{\text{rad}} = \Delta - \frac{\Delta-1}{\text{rad}} \geq \max\{\alpha_\ell = \Delta - \frac{\Delta-1}{\text{rad}} \mid \ell \in [2, \text{rad}]\}$. We show that $\beta_{\text{rad}+1} \leq \alpha_{\text{rad}}$.

$$
\begin{aligned}
\beta_{\text{rad}+1} = \Delta - \frac{\Delta - 1}{\text{rad}+1} - \frac{\lfloor\frac{\Delta}{2}\rfloor}{\text{rad}+1} \leq \Delta - \frac{\Delta-1}{\text{rad}} &= \alpha_{\text{rad}} \\
\Leftrightarrow \frac{\Delta-1}{\text{rad}} \leq \frac{\Delta-1}{\text{rad}+1} + &\frac{\lfloor\frac{\Delta}{2}\rfloor}{\text{rad}+1} \\
\Leftrightarrow (\text{rad}+1)\cdot(\Delta-1) \leq \text{rad}(\Delta-1) + &\text{rad}\lfloor\frac{\Delta}{2}\rfloor \\
\Leftrightarrow \Delta - 1 \leq \text{rad}\lfloor\frac{\Delta}{2}\rfloor
\end{aligned}
$$

This holds true, since $\text{rad} \geq 2$.

Hence, if $\text{diam} = \text{rad} + 1$, the statement holds. Otherwise, we still have to show that $\alpha_{\text{rad}} \geq \max\{\alpha_\ell \mid \text{rad} + 2 \leq \ell \leq \text{diam}\}$. As shown in the proof of Lemma 6, $\alpha_{\text{rad}+2} \geq \max\{\alpha_\ell \mid \text{rad}+2 \leq \ell \leq \text{diam}\}$, and $\alpha_{\text{rad}} \geq \alpha_{\text{rad}+2}$. Hence, the total stretch is at most $\Delta - \frac{\Delta-1}{\text{rad}} = \alpha_{\text{rad}} \geq \max\{\alpha_\ell \mid 2 \leq \ell \leq \text{diam}, \ell \neq \text{rad}+1\} \cup \{\beta_{\text{rad}+1}\}$. □

As we will show in Section 5, there are instances, where this stretch is achieved by the radius algorithm and it is NP-hard to decide whether a better stretch is possible. This then shows that this simple algorithm produces the best possible stretch for some instances in polynomial time, unless P = NP.

Finally, we can show that the radius algorithm performs well if the input graph is a tree.

Lemma 8. *Let $(G = (V, E), \Delta)$ be an instance of* SPTGR *where G is a tree. Then the radius algorithm computes a labeling with stretch at most $\frac{\Delta+1}{2}$.*

Proof. Similar to the previous proofs, we give for each distance $\ell \in [2, \text{diam}]$ an upper bound α_ℓ for the stretch of distance-ℓ vertex pairs.

Let u, v be a vertex pair of distance-ℓ. We make the following case distinction. Consider the case where u is an ancestor of v or v is an ancestor of u. In both cases, the edges of the fastest paths from u to v are alternatively labeled with Δ and $\lceil\frac{\Delta}{2}\rceil$ by the algorithm. Hence, we have that the duration of the fastest path from u to v is at most $\frac{\ell-2}{2}\cdot\Delta + \lfloor\frac{\Delta}{2}\rfloor + 1$ if ℓ is even, and at most $\frac{\ell-1}{2}\cdot\Delta + 1$ if ℓ is odd. Hence, the stretch is $\alpha \leq \frac{\Delta}{2} + \frac{1}{\ell}$. Since $\ell \geq 2$, we have that $\alpha \leq \frac{\Delta+1}{2}$.

Now consider the case that neither u is an ancestor of v, nor is v an ancestor of u. Let w be the closest common ancestor of v and u. Then we have that a fastest temporal path from u to v can be decomposed into a fastest temporal path from u to w and a fastest temporal path from w to v. Note that the waiting time at w is Δ, since all edges from w to its children have the same label. All other waiting times are $\lfloor \frac{\Delta}{2} \rfloor$ and $\lceil \frac{\Delta}{2} \rceil$, alternatingly. It follows that the duration of a fastest temporal path from u to v is at most $\frac{\ell-1}{2} \cdot \Delta + \lfloor \frac{\Delta}{2} \rfloor + 1$ if ℓ is even, and at most $\frac{\ell}{2} \cdot \Delta + 1$ if ℓ is odd. Then, again, we have that the stretch is $\alpha = \frac{\Delta}{2} + \frac{1}{\ell}$. Since $\ell \geq 2$, we have that $\alpha \leq \frac{\Delta+1}{2}$. □

On trees with a large maximum degree, we can show that our algorithm is optimal.

Lemma 9. *The radius algorithm computes the optimum stretch for trees with maximum degree at least* $\Delta + 1$ *and it is a 2-approximation algorithm on general trees.*

Proof. Let G be a tree with maximum degree at least $\Delta + 1$, and let v be a vertex of G where $\deg(v) \geq \Delta + 1$.[8] Then, since every edge of G gets one label between 1 and Δ, in any labeling there must be at least two neighbors u_1, u_2 of v such that the edges u_1v and u_2v get the same label. Then the fastest path from u_1 to u_2 in this labeling has duration $\Delta + 1$, while the shortest path between u_1, u_2 has length 2. Therefore, the stretch of G is at least $\frac{\Delta+1}{2}$, and thus the radius algorithm provides the optimum stretch by Lemma 8.

Now let G be an arbitrary tree. Let v be an arbitrary non-leaf vertex of G, and let u_1, u_2 be any two of its neighbors. Let λ_{OPT} be a labeling that achieves an optimum stretch for G, and let ℓ_1 (resp. ℓ_2) be the label of the edge u_1v (resp. u_2v) in λ_{OPT}. We will prove a lower bound for the stretch of λ_{OPT}. First note that, if $\ell_1 = \ell_2$ then the durations of the fastest temporal paths from u_1 to u_2 and from u_2 to u_1 are both $\Delta + 1$, while both these durations are strictly less than $\Delta + 1$ if $\ell_1 \neq \ell_2$. Thus let us assume that $\ell_1 \neq \ell_2$, and let without loss of generality $\ell_1 < \ell_2$. Then the fastest temporal path from u_1 to u_2 (resp. from u_2 to u_1) is $\ell_2 - \ell_1 + 1$ (resp. $\Delta + \ell_1 - \ell_2 + 1$), while the distance of the shortest path between u_1 and u_2 is 2. Then $\max\{\ell_2 - \ell_1 + 1, \Delta + \ell_1 - \ell_2 + 1\}$ is minimized when $\ell_2 = \ell_1 + \frac{\Delta}{2}$ (when Δ is even) or when $\ell_2 = \ell_1 + \frac{\Delta+1}{2}$ (when Δ is odd), and in this case the duration of the fastest temporal paths both from u_1 to u_2 and from u_2 to u_1 become at least $\frac{\Delta}{2} + 1 = \frac{\Delta+2}{2}$. Therefore, the optimum stretch

[8] Recall that Δ is the period of the Δ-periodic temporal graph we aim to construct and not the maximum degree of the input graph.

in λ_{OPT} is at least $\frac{\Delta+2}{4}$. Thus, by Lemma 8, the radius algorithm returns an approximation of the stretch with ratio at most $\frac{\frac{\Delta+1}{2}}{\frac{\Delta+2}{4}} < 2$. $\square$

5. General Hardness Results

In this section, we present NP-hardness results for SPTGR for (i) all constant values of $\Delta \geq 3$ and (ii) all constant values of $\alpha \geq 1$. All of our results are achieved by reductions from 3-Coloring, which is NP-hard [30].

> 3-Coloring
> Input: A graph $G = (V, E)$.
> Question: Is there a *proper* 3*-coloring* χ of G, that is, a function $\chi\colon V \rightarrow \{1, 2, 3\}$, such that for each edge $\{u, v\} \in E$, $\chi(u) \neq \chi(v)$?

First, we will present two hardness results for $\Delta = 3$: one for $\alpha \in [1, 1.5)$ and one for $\alpha \in [1.5, 2)$. The first reduction answers an open question by Erlebach et al. [23] and the second reduction proves that our presented radius algorithm is tight on some hard instances. That is, we show that on the build instances, the radius algorithm is guaranteed to achieve a stretch of 2 and it is NP-hard to decide whether a better stretch is possible on these instances. Afterwards, we present hardness results for all other constant values of $\Delta \geq 4$ and $\alpha \geq 2$. To this end, we will adapt the latter reduction for $\Delta = 3$ by replacing parts of the instance by some gadgets that we define for each $\Delta > 3$.

5.1. Hardness for $\Delta = 3$

We now start by presenting our first hardness result.

Lemma 10. *For each $\alpha \in [1, 1.5)$,* SPTGR *is NP-hard even if $\Delta = 3$ and G has diameter* 2*.*

Proof. We reduce from 3-Coloring. Let $G = (V, E)$ be an instance of 3-Coloring and assume that each vertex of V has at least one non-neighbor. Moreover, let $\alpha \in [1, 1.5)$. We obtain an instance $I' := (G' = (V', E'), \Delta = 3, \alpha)$ of SPTGR as follows: We initialize the graph $G' := (V', E')$ as the star with center vertex c and leaf set V. We add another vertex c^* which we make adjacent to all vertices. Additionally, we add for each non-edge $\{u, v\}$ of G the vertices $x_{u,v}$ and $x_{v,u}$, and the edges $\{u, x_{u,v}\}, \{u, x_{v,u}\}, \{v, x_{u,v}\}, \{v, x_{v,u}\}$, and $\{x_{u,v}, x_{v,u}\}$ to G'. That is, $G'[\{u, v, x_{u,v}, x_{v,u}\}]$ is a diamond. Finally, we

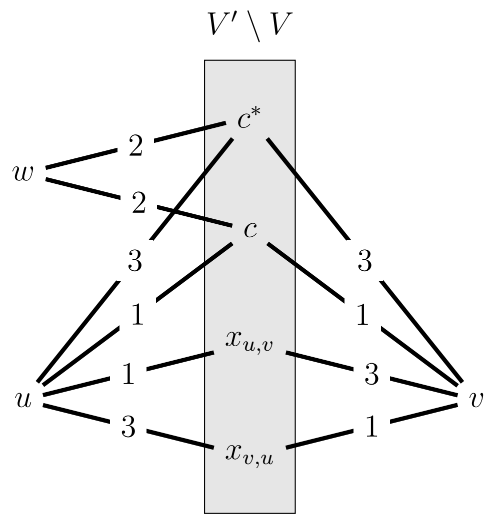

Figure 2: An illustration for the main aspects of the reduction behind Lemma 10. Here, u, v, and w are distinct vertices from the 3-Coloring instance with vertex set V in which u and v are not adjacent but w is adjacent to both u and v. Recall that $V' \setminus V$ is a clique. The depicted labeling achieves a stretch of 1, where the non-depicted edges of the clique $V' \setminus V$ all receive label 2.

turn $\{c^*, c\} \cup (V' \setminus V)$ into a clique in G'. This completes the construction of I'. Figure 2 shows illustrated the main aspects of the construction. Note that G' has a diameter of 2, since G' is a split graph where both vertices c and c^* are adjacent to all other vertices of the graph. In the following, let D be the distance matrix of G'. The intuition of this reduction is that for each edge $\{u, v\}$ of G, (u, c, v) and (u, c^*, v) are the only paths of length less than $\Delta > 2 \cdot \alpha = \alpha \cdot D_{u,v}$. Hence, on both paths, the labels of both edges need to be distinct in each labeling λ of stretch at most α. Next, we show that G is 3-colorable if and only if I is a yes-instance of SPTGR.

($\Rightarrow$) Let $\chi\colon V \to \{1, 2, 3\}$ be a 3-coloring of G. We define an edge labeling λ of G' of stretch 1 as follows: For each vertex $v \in V$, we set $\lambda(\{c, v\}) := \chi(v)$ and $\lambda(\{c^*, v\}) := 4 - \chi(v)$. For each non-edge $\{u, v\}$ of G, we set $\lambda(\{u, x_{u,v}\}) := \lambda(\{v, x_{v,u}\}) := 1$ and $\lambda(\{u, x_{v,u}\}) := \lambda(\{v, x_{u,v}\}) := 3$. For all other edges e of G', we set $\lambda(e) := 2$. Note that these latter edges are the edges of the clique $V' \setminus V$. We now show that λ has a stretch of 1, that is, for each pair of vertices of distance 2 in G', there are temporal paths of duration 2 between them. Let (a, b) be a pair of vertices of distance 2 in G'. Since $V' \setminus V$ is a clique in G', at least one of a and b is a vertex of V. Without loss of generality assume that $a \in V$. We distinguish two cases.

First, assume that $b \notin V$. By construction of G' this implies that $b = x_{u,v}$ for some non-edge u, v of G with $a \notin \{u, v\}$. Since we assumed that each vertex of V has at least one non-neighbor in G, there is a vertex $w \in V$ such that the vertices $x_{a,w}$ and $x_{w,a}$ exist. Hence, by definition of λ, the path $(a, x_{a,w}, x_{u,v} = b)$ uses the labels 1 and 2, and the path $(b = x_{u,v}, x_{w,a}, a)$ uses the labels 2 and 3. Consequently, there are temporal paths between a and b of durations 2.

Second, assume that $b \in V$. If $\{a, b\}$ is a non-edge of G, then the vertices $x_{a,b}$ and $x_{b,a}$ exist. Hence, by definition of λ, the paths $(a, x_{b,a}, b)$ and $(b, x_{a,b}, a)$ uses the labels 3 and 1. Consequently, there are temporal paths between a and b of durations 2. Otherwise, that is, if $\{a, b\}$ is an edge of G, then $\chi(a) \neq \chi(b)$ since χ is a proper coloring of G. Since $\Delta = 3$, this implies that $\chi(b) = \chi(a) + 1$ (modulo Δ) or that $\chi(a) = \chi(b) + 1$ (modulo Δ). Assume without loss of generality that the first is the case. Hence, by definition of λ, the path (a, c, b) uses consecutive time labels (modulo Δ) and the path (b, c^*, a) uses consecutive time labels (modulo Δ). Consequently, there are temporal paths between a and b of durations 2. Concluding, λ has a stretch of $1 \leq \alpha$, which implies that I' is a yes-instance of SPTGR.

($\Leftarrow$) Let $\lambda\colon E' \to \{1, 2, 3\}$ be an edge labeling of G' with stretch at most α. We define a 3-coloring χ of the vertices of V as follows: For each vertex $v \in V$, we set $\chi(v) := \lambda(\{c, v\})$. Next, we show that for each edge $\{u, v\} \in E$, u and v receive distinct colors under χ. Recall that for λ to have a stretch of α for G', at least one path of length at most $\alpha \cdot D_{u,v} = \alpha \cdot D_{v,u} = \alpha \cdot 2 < 3$ from u to v in G' has duration at most $\alpha \cdot 2 < 3$. Since $\{u, v\}$ is an edge of E, the vertices $x_{u,v}$ and $x_{v,u}$ do not exist, which implies that (u, c, v) and (u, c^*, v) are the only paths from u to v in G' of length less than 3 and that (v, c, u) and (v, c^*, u) are the only paths from v to u in G' of length less than 3. Assume towards a contradiction that $\chi(u) = \chi(v)$. This implies that $\lambda(\{c, u\}) = \lambda(\{c, v\})$. Hence, the paths (u, c, v) and (v, c, u) have a duration of exactly $\Delta + 1 = 4 > 2 \cdot \alpha = \alpha \cdot D_{u,v} = \alpha \cdot D_{v,u}$. Consequently, for λ to have a stretch of α for G', both paths (u, c^*, v) and (v, c^*, u) must have a duration of 2. Since $\Delta = 3$, this is impossible. This contradicts the assumption that λ is an edge labeling of G' with stretch at most α. Thus, $\chi(u) = \lambda(\{c, u\}) \neq \lambda(\{c, v\}) = \chi(v)$, which implies that χ is a proper 3-coloring for G'. □

Note that Lemma 10 answers (by setting $\alpha = 1$) an open question by

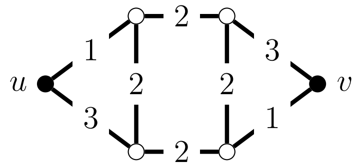

Figure 3: The sunglasses gadgets for $\Delta = 3$ for the vertices u and v with the sunglasses labeling. The black vertices are the docking points and the white vertices are the central vertices.

Erlebach et al. [23] about the complexity of finding a periodic Δ-labeling for a diameter-2 graph G, such that the fastest paths between any two vertices of G equals their distance.

Next, we show that for each $\Delta \geq 3$, it is NP-hard to decide whether a stretch in $[\frac{\Delta}{2}, \frac{\Delta+1}{2})$ can be achieved. To this end, we define gadget graphs for each Δ and some associated labelings. First, we define this gadgets for $\Delta = 3$ and present the first reduction. Afterwards, we define these gadgets for odd values of $\Delta > 3$ and even values of $\Delta \geq 4$ and present similar reductions. The gadgets and their respective labeling for $\Delta \in [3, 10]$ can be seen in Figures 3 to 5.

Definition 11 (Sunglasses gadgets for $\Delta = 3$)**.** *A* 3-sunglasses gadget with docking points u and v *is the graph shown in Figure 3, where the black vertices indicate the docking points u and v, and the white vertices are* central vertices.

Figure 3 also shows what we call the *sunglasses labeling*.

Definition 12. *Let $\Delta > 1$, let $G = (V, E)$ be a graph, and let $\lambda \colon E \to [1, \Delta]$. Moreover, let u and v be vertices of distance exactly 2 in G. We call a vertex z of G a* nice neighbor of u and v, *if (i) z is a common neighbor of u and v and (ii) $\lambda(\{u, z\}) \neq \lambda(\{v, z\})$.*

Observation 13. *Let $\Delta > 1$, let $G = (V, E)$ be a graph, and let $\lambda \colon E \to [1, \Delta]$. Moreover, let u and v be vertices of distance exactly 2 in G. If u and v have a nice neighbor z, then (u, z, v) and (v, z, u) are paths of duration at most Δ each.*

We now show that there are graphs with diameter 3 and radius 2, for which it is NP-hard to decide whether one can achieve a better stretch than the one achieved by the radius algorithm. This then shows that this simple and natural algorithm is tight on some hard instances.

Lemma 14. *For $\Delta = 3$ and each $\alpha \in [1.5, 2)$,* SPTGR *is NP-hard even on graphs with diameter* 3*, radius* 2*, and where the radius algorithm produces a labeling of stretch* 2*.*

Proof. Let $\Delta = 3$ and let $\alpha \in [1.5, 2)$. We again reduce from 3-Coloring. Let $G = (V, E)$ be an instance of 3-Coloring. Again, assume that each vertex of V has at least one non-neighbor, as otherwise we would know that this vertex receives a unique color in any 3-coloring and the remaining vertices can only be colored in two colors, which can be checked in polynomial time. We obtain an equivalent instance $I := (G', \Delta, \alpha)$ of SPTGR as follows: We initialize the graph $G' := (V', E')$ as the star with center vertex c and leaf set V. Additionally, we add for each non-edge $\{u, v\}$ of G a 3-sunglasses gadget $S_{\{u,v\}}$ with docking points u and v to G'. Let X denote the set of the central vertices of all added sunglasses gadgets.

To complete the definition of G', we add the vertices $\widehat{X} := \{\widehat{x} \mid x \in X\}$ to G' and making each vertex of $\widehat{X}$ adjacent to each vertex of $X \cup \widehat{X} \cup \{c\}$ in G'.

This completes the construction of I. In the following, let D be the distance matrix of G'. The intuitive idea is that for each edge $\{u, v\} \in E$, the path (u, c, v) $((v, c, u))$ is the only path of length less than $\Delta + 1 = 4 > \alpha \cdot D_{u,v} = 2 \cdot \alpha$ from u (v) to v (u) in G'. Hence, each labeling λ of G' of stretch at most α has to assign distinct labels to the edges $\{c, u\}$ and $\{c, v\}$. In other words, if labeling λ of G' has stretch at most α, then the edges between c and the vertices of V imply a 3-coloring for G.

Structural Properties. Note that the radius of G' is at most 2: The neighborhood of vertex c is $V \cap \widehat{X}$ and each vertex of $X = V' \setminus (V \cup \widehat{X} \cup \{c\})$ is a neighbor of each vertex of $\widehat{X}$. Moreover, G' has a diameter of 3, since (i) each vertex of V has distance at most 2 to each vertex of $V \cup \widehat{X}$ by going over c and thus distance at most 3 to each vertex of X and (ii) each vertex of X has distance at most 2 to each vertex of $X \cup \{c\}$ by going over some vertex of $\widehat{X}$ and thus distance at most 3 to each vertex of V. In particular, all vertex pairs of distance exactly 3 in G' contain one vertex of V and one vertex of X. Since c is a neighbor of all vertices of V, G' fulfills the property of Lemma 7, which implies that the radius algorithm produces a labeling of stretch of at most $\Delta - \frac{\Delta - 1}{\mathrm{rad}} = 2$.

Correctness. We now show that G is 3-colorable if and only if I is a yes-instance of SPTGR.

($\Leftarrow$) Let $\lambda\colon E' \to \{1,2,3\}$ be an edge labeling of G' with stretch at most α. We define a 3-coloring $\chi\colon V \to \{1,2,3\}$ as follows: For each vertex $v \in V$, we set $\chi(v) := \lambda(\{c,v\})$. Next, we show that for each edge $\{u,v\} \in E$, u and v receive distinct colors under χ. Since $\{u,v\}$ is an edge of E, there is no sunglasses gadget with docking points u and v in G'. This implies that (u,c,v) is the only path from u to v in G' of length less than $\Delta + 1 = 4 > \alpha \cdot D_{u,v} = 2 \cdot \alpha$. Assume towards a contradiction that $\chi(u) = \chi(v)$. This would imply that $\lambda(\{c,u\}) = \lambda(\{c,v\})$. Hence, the unique path (u,c,v) from u to v in G' of length less than $\Delta + 1 = 4$ has a duration of exactly $4 = \Delta + 1 > 2 \cdot \alpha = \alpha \cdot D_{u,v}$. This contradicts the assumption that λ is an edge labeling of G' with stretch at most α. Thus, $\chi(u) = \lambda(\{c,u\}) \neq \lambda(\{c,v\}) = \chi(v)$, which implies that χ is a proper 3-coloring for G'.

($\Rightarrow$) Let $\chi\colon V \to \{1,2,3\}$ be a 3-coloring of G. Assume without loss of generality that each of the three colors is assigned at least once. We define an edge labeling λ of G' of stretch $\frac{\Delta}{2} = 1.5$ as follows: For each vertex $v \in V$, we set $\lambda(\{c,v\}) := \chi(v)$. For each non-edge $\{a,b\}$ of G (that is, for each added sunglasses gadget), we label the edges of the sunglasses gadget $S_{x,y}$ according the sunglasses labeling (see Figure 3). To complete the definition of λ, it remains to define the labels of the edges incident with the vertices of $\widehat{X}$. For each vertex $\widehat{x} \in \widehat{X}$, we set $\lambda(\{x,\widehat{x}\}) := 1$. For each other edge e incident with at least one vertex of $\widehat{X}$, we set $\lambda(e) := 2$.

This completes the definition of λ. Next, we show that λ has a stretch of $1.5 = \frac{\Delta}{2} \leq \alpha$.

First, we consider vertex pairs $\{y,z\}$ of distance 2 in G'. For these vertex pairs, we show that there are paths of duration at most $3 = 1.5 \cdot D_{y,z}$ between y and z.

- If $y = c$, then by the initial argumentation, $z \in X$. Hence, $\widehat{z}$ is a nice neighbor of c and z, which implies that the path $(c, \widehat{z}, z)$ has duration at most $\Delta = 3$ in both directions.

- If $y \in \widehat{X}$, then by the initial argumentation, $z \in V$. Since z is the docking point of at least one sunglasses gadget, there are at least two vertices x_1 and x_2 of X adjacent to z. For at least one $i \in \{1,2\}$, $z \neq \widehat{x_i}$. Hence, the edge $\{x_i, z\}$ receives label 2 under λ. This implies that x_i is a nice neighbor, since the edge $\{y, x_i\}$ receives label either 1 or 3 under λ. Consequently, the path $(c, \widehat{z}, z)$ has duration at most $\Delta = 3$ in both directions.

- If $y \in X$, then $z \in X \cup V \cup \{c\}$. The case for $z = c$ is covered by the above cases. If $z \in X$, the path $(y, \widehat{y}, z)$ has labels $(1, 2)$ and thus has duration at most $\Delta = 3$ in both directions. Otherwise, if $z \in V$, then by construction, z is a docking point of the sunglasses gadget that contains y. Hence, by the sunglasses labeling, there is a nice neighbor of y and z in this sunglasses gadget which implies the existence of paths of duration at most 3 between in y and z.

- If $y \in V$, then $z \in X \cup \widehat{X} \cup V$. The case for $z \in X \cup \widehat{X}$ is covered by the above cases. If $z \in V$, we consider two cases. If $\{y, z\}$ is an edge of E, c is a nice neighbor of y and z, since χ is a proper 3-coloring of G. Otherwise, if $\{y, z\}$ is not an edge of G, there is a sunglasses gadget with docking points y and z. By the sunglasses labeling, this gadget contains a path with labels $(1, 2, 3)$ from y to z and a path with labels $(1, 2, 3)$ from z to y. In both cases, there are paths of duration at most 3 between y and z.

Next, we consider vertex pairs $\{y, z\}$ of distance 3 in G'. For these vertex pairs, we show that there are paths of duration at most $4 < 4.5 = 1.5 \cdot D_{y,z}$ between y and z. By the initial argumentation about the structural properties of G', we can assume without loss of generality that $y \in V$ and $z \in X$. Moreover, z is not part of any sunglasses gadget attached to y, as otherwise, the distance between y and z would be at most 2. Take an arbitrary sunglasses gadget attached to y and let x_1 and x_3 be the neighbors of y in this sunglasses gadget. By the sunglasses labeling, we can assume that the label of $\{y, x_i\}$ is equal to i for each $i \in \{1, 3\}$. Since $z \in X$, $\widehat{z}$ is a vertex of $\widehat{X}$, $\lambda(\{z, \widehat{z}\}) = 1$, and $\lambda(\{\widehat{z}, x_1\}) = \lambda(\{\widehat{z}, x_3\}) = 2$. This implies that the path $(z, \widehat{z}, x_3, y)$ has labels $(1, 2, 3)$ and thus a duration of 3. For the other direction, the path $(y, x_1, \widehat{z}, z)$ has labels $(1, 2, 1)$ and thus a duration of 4. Hence, there are paths of duration at most 4 between y and z.

This implies that the stretch of λ is at most $\frac{\Delta}{2} = 1.5 \leq \alpha$, which completes the proof. □

In combination with Lemma 10, this implies that for $\Delta = 3$, SPTGR is NP-hard for each $\alpha \in [1, 2)$.

Corollary 15. *For $\Delta = 3$, SPTGR is NP-hard for each $\alpha \in [1, 2)$.*

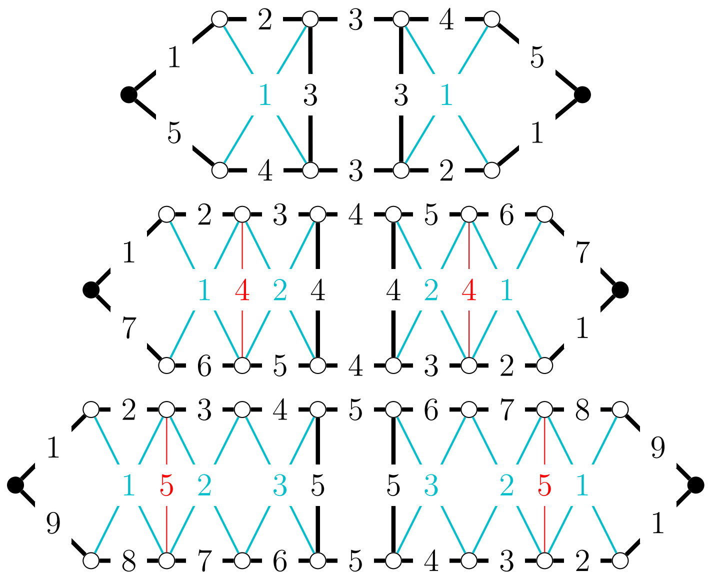

Figure 4: The sunglasses gadgets for odd $\Delta > 3$. The black vertices indicate the docking points, the black edges indicate the edges of the chronological paths and cycles, the teal edges indicate the four zigzag paths and the red edges are the parallel edges. For $\Delta = 5$, the parallel edges coincide with edges of the chronological cycles. The shown labeling is the *sunglasses labeling*. Formally, this labeling labels (a) the chronological paths increasingly from 1 to Δ, (b) the zigzag paths increasingly from 1 to $\frac{\Delta-2}{2}$, and (c) all other edges (namely, all vertically drawn edges) with label $\frac{\Delta+1}{2}$. Note that in this way, both chronological cycles are assigned increasing labels from 1 to Δ.

5.2. Hardness for $\Delta > 3$

We now proceed by showing that also for each $\Delta > 3$, SPTGR is NP-hard for each $\alpha \in [\frac{\Delta}{2}, \frac{\Delta+1}{2})$. To obtain this result, we define for each $\Delta > 4$, Δ-sunglasses gadgets and replace the 3-sunglasses gadgets in the reduction of the proof of Lemma 14 by the larger Δ-sunglasses gadgets. The gadgets are slightly different for odd and even values of Δ. We first define the gadgets for odd Δ. See Figure 4 for an illustration.

Definition 16 (Sunglasses gadgets for odd $\Delta > 3$)**.** *Let Δ be an odd integer with $\Delta > 3$ and let $e := \{u, v\}$. A* Δ-sunglasses gadget with docking points u and v *is the graph $G = (V, E)$, consisting of a cycle $(u = p_e^{u,0}, p_e^{u,1}, \ldots, p_e^{u,\Delta-1}, p_e^{u,\Delta} = v = p_e^{v,0}, p_e^{v,1}, \ldots, p_e^{v,\Delta-1}, p_e^{v,\Delta} = u)$ of length $2 \cdot \Delta$ with several shortcuts that we define in the following. The shortcuts are as follows:*

- *the edges* $\{p_e^{u,\frac{\Delta-1}{2}}, p_e^{v,\frac{\Delta+1}{2}}\}$ *and* $\{p_e^{v,\frac{\Delta-1}{2}}, p_e^{u,\frac{\Delta+1}{2}}\}$*,*
- *the* parallel edges $\{p_e^{u,2}, p_e^{v,\Delta-2}\}$ *and* $\{p_e^{v,2}, p_e^{u,\Delta-2}\}$*, and*
- *for each* $i \in [1, \frac{\Delta-3}{2}]$*, the* zigzag edges $\{p_e^{u,i}, p_e^{v,\Delta-i-1}\}$, $\{p_e^{u,i+1}, p_e^{v,\Delta-i}\}$, $\{p_e^{v,i}, p_e^{u,\Delta-i-1}\}$*, and* $\{p_e^{v,i+1}, p_e^{u,\Delta-i}\}$*.*

Let $a \in \{u,v\}$ *and let* $b \in \{u,v\} \setminus \{a\}$*. We call* $P_e^a = (a = p_e^{a,0}, p_e^{a,1}, \dots, p_e^{a,\Delta-1}, p_e^{a,\Delta} = b)$ *the* chronological (a,b)-path *in* G*. Moreover, we call the cycle* $C_e^a = (a, p_e^{a,1}, \dots, p_e^{a,\frac{\Delta-1}{2}}, p_e^{b,\frac{\Delta+1}{2}}, \dots p_e^{b,\Delta-1}, a)$ *the* chronological a-cycle *in* G*. The vertices* $p_e^{u,\frac{\Delta-1}{2}}$*,* $p_e^{u,\frac{\Delta+1}{2}}$*,* $p_e^{v,\frac{\Delta-1}{2}}$*, and* $p_e^{v,\frac{\Delta+1}{2}}$ *are the* central vertices *of* G*. Finally,* G *contains four* zigzag paths*. These are the unique paths that start in some vertex of* $\{p_e^{u,1}, p_e^{u,\Delta-1}, p_e^{v,1}, p_e^{v,\Delta-1}\}$*, only use zigzag edges, and end in a central vertex.*

Figure 4 also defines the *sunglasses labeling* for these gadgets. Next, we define the gadgets for even values of Δ. See Figure 5 for an illustration. Roughly speaking, for the case that Δ is even, the Δ-sunglasses gadgets are again cycles of length $2 \cdot \Delta$ with several shortcuts. The main difference is, that there are only two central vertices, namely the two vertices of half distance between the docking points. The chronological cycles do not use an edge between these central vertices anymore but rather use a parallel edge between their neighbors. That is, the chronological cycles have length only $\Delta - 1$ in the case where Δ is even.

Definition 17 (Sunglasses gadgets for even $\Delta \geq 4$)**.** *Let* Δ *be an even integer with* $\Delta \geq 4$ *and let* $e := \{u,v\}$*. A* Δ-sunglasses gadget with docking points u and v *is the graph* $G = (V,E)$*, consisting of a cycle* $(u = p_e^{u,0}, p_e^{u,1}, \dots, p_e^{u,\Delta-1}, p_e^{u,\Delta} = v = p_e^{v,0}, p_e^{v,1}, \dots, p_e^{v,\Delta-1}, p_e^{v,\Delta} = u)$ *of length* $2 \cdot \Delta$ *with several shortcuts that we define in the following. The shortcuts are as follows:*

- *the edges* $\{p_e^{u,\frac{\Delta}{2}-1}, p_e^{v,\frac{\Delta}{2}+1}\}$ *and* $\{p_e^{v,\frac{\Delta}{2}-1}, p_e^{u,\frac{\Delta}{2}+1}\}$*,*
- *the* parallel edges $\{p_e^{u,2}, p_e^{v,\Delta-2}\}$ *and* $\{p_e^{v,2}, p_e^{u,\Delta-2}\}$*, and*
- *for each* $i \in [1, \frac{\Delta}{2}-1]$*, the* zigzag edges $\{p_e^{u,i}, p_e^{v,\Delta-i-1}\}$, $\{p_e^{u,i+1}, p_e^{v,\Delta-i}\}$, $\{p_e^{v,i}, p_e^{u,\Delta-i-1}\}$*, and* $\{p_e^{v,i+1}, p_e^{u,\Delta-i}\}$*.*

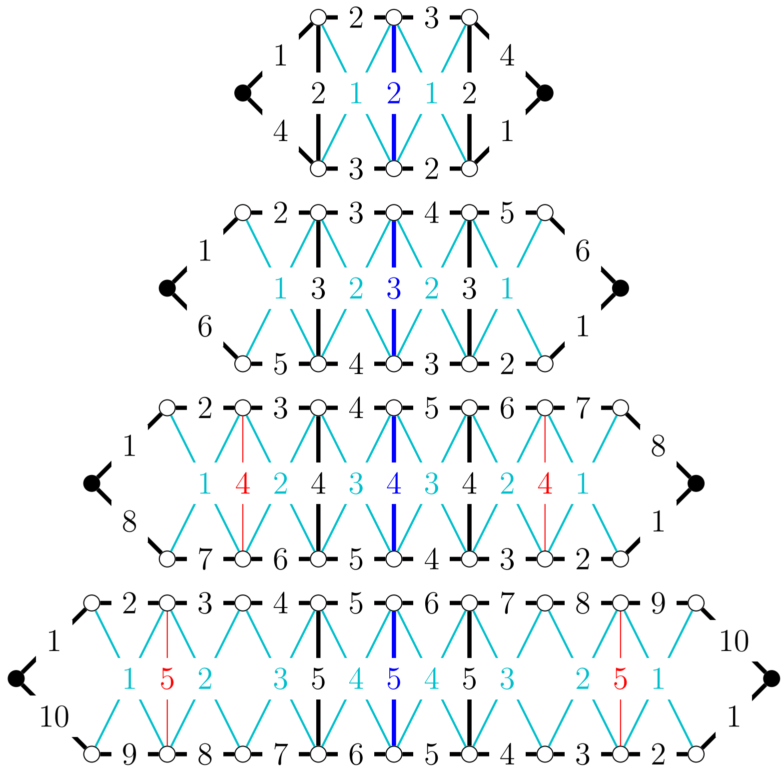

Figure 5: The sunglasses gadgets for even $\Delta \geq 4$. Again, the black vertices indicate the docking points, the black edges indicate the edges of the chronological paths and cycles, the teal edges indicate the four zigzag paths and the red edges are the parallel edges. For $\Delta \in \{4, 6\}$, the parallel edges coincide with edges of the chronological cycles, or the edge (indicated in blue) between the central vertices. The latter described edge is formally not part of the sunglasses gadget for $\Delta > 4$ but is added in the NP-hardness reduction anyway. The shown labeling is the *sunglasses labeling*. Formally, this labeling labels /a) the chronological paths increasingly from 1 to Δ, (b) the zigzag paths increasingly from 1 to $\frac{\Delta}{2} - 1$, and (c) all other edges (namely, all vertically drawn edges) with label $\frac{\Delta}{2}$. Note that in this way, both chronological cycles are assigned strictly increasing labels.

Let $a \in \{u,v\}$ and let $b \in \{u,v\} \setminus \{a\}$. We call $P_e^a = (a = p_e^{a,0}, p_e^{a,1}, \dots, p_e^{a,\Delta-1}, p_e^{a,\Delta} = b)$ the chronological (a,b)-path *in G. Moreover, we call the cycle $C_e^a = (a, p_e^{a,1}, \dots, p_e^{a,\frac{\Delta}{2}-1}, p_e^{b,\frac{\Delta}{2}+1}, \dots p_e^{b,\Delta-1}, a)$ the* chronological a-cycle *in G. The vertices $p_e^{u,\frac{\Delta}{2}}$ and $p_e^{v,\frac{\Delta}{2}}$ are the* central vertices *of G. Finally, G contains four* zigzag paths*. These are the unique paths that start in some vertex of $\{p_e^{u,1}, p_e^{u,\Delta-1}, p_e^{v,1}, p_e^{v,\Delta-1}\}$, only use zigzag edges, and end in a central vertex.*

Similar to the odd case, Figure 5 also defines the *sunglasses labeling* for these gadgets. With these defined gadgets, we now present our hardness result.

Theorem 18. *For each $\Delta \geq 4$ and each $\alpha \in [\frac{\Delta}{2}, \frac{\Delta+1}{2})$,* SPTGR *is NP-hard on graphs of diameter $\mathcal{O}(\Delta)$.*

Construction. Let $\Delta \geq 4$ and let $\alpha \in [\frac{\Delta}{2}, \frac{\Delta+1}{2})$. We again reduce from 3-Coloring. Let $G = (V, E)$ be an instance of 3-Coloring. Again, assume that each vertex of V has at least one non-neighbor, as otherwise we would know that this vertex receives a unique color in any 3-coloring and the remaining vertices can only be colored in two colors, which can be checked in polynomial time. We obtain an equivalent instance $I := (G', \Delta, \alpha)$ of SPTGR as follows: We initialize the graph $G' := (V', E')$ as the star with center vertex c and leaf set $V \cup V^*$, where $V^* := \{v_i^* \mid i \in [1, \Delta - 3]\}$. Additionally, we add for each non-edge $\{u,v\}$ of G a Δ-sunglasses gadget $S_{\{u,v\}}$ with docking points u and v to G'. Let X denote the set of the central vertices of all added sunglasses gadgets and let Y denote the set of all other internal vertices of all added sunglasses gadgets.

To complete the definition of G', we distinguish between three non-exclusive cases:

- If Δ is even, we turn the set X of all central vertices into a clique in G'.
- If $\Delta = 4$, we additionally add three vertices c_1, c_2, and c_3 to G' and add edges, such that $N_{G'}[c_i] = \{c\} \cup X \cup \{c_1, c_2, c_3\}$ for each $i \in \{1, 2, 3\}$. This case is depicted in Figure 6.
- If Δ is odd, we add the vertices $\widehat{X} := \{\widehat{x} \mid x \in X\}$ to G' and make each vertex of $\widehat{X}$ adjacent to each other vertex of $X \cup \widehat{X}$ in G'.

This completes the construction of I. In the following, let D be the distance matrix of G'.

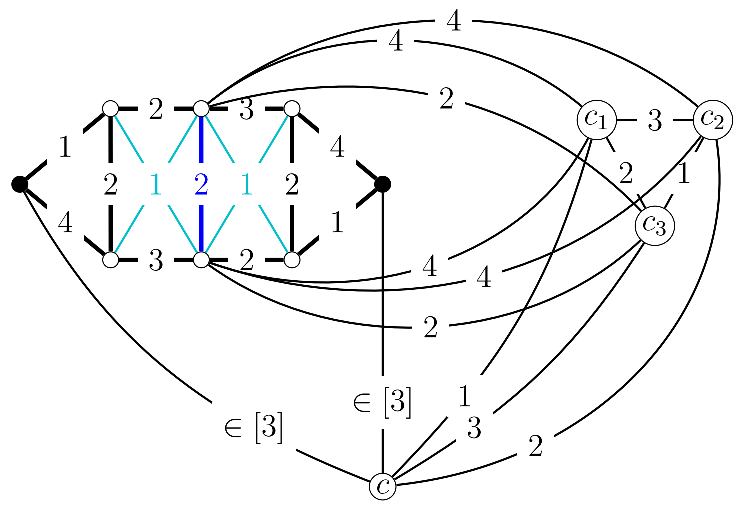

Figure 6: An illustration of the additional vertices $\{c_1, c_2, c_3\}$ and edges (as well as their labels) for the reduction in the special case of $\Delta = 4$.

Intuition. The intuitive idea is that for each edge $\{u, v\} \in E$, the path (u, c, v) (resp. (v, c, u)) is the only path of length less than $\Delta + 1 > \alpha \cdot D_{u,v} = 2 \cdot \alpha$ from u to v (from v to u) in G'. Hence, each labeling λ of G' of stretch at most α has to assign distinct labels to the edges $\{c, u\}$ and $\{c, v\}$. In other words, if labeling λ of G' has stretch at most α, then the edges between c and the vertices of V imply a Δ-coloring for G. From these Δ colors, only 3 can actually be assigned to the vertices of V, since each of the edges from c to vertices of V^* will receive a unique label. We now show that G is 3-colorable if and only if I is a yes-instance of SPTGR.

Correctness. First, we show the forwards direction.

Lemma 19. *G is 3-colorable if I is a yes-instance of* SPTGR

Proof. Let $\lambda \colon E' \to [1, \Delta]$ be an edge labeling of G' with stretch at most α. First, we show that there is a set $L' \subseteq [1, \Delta]$ of size at most three, such that for each vertex $v \in V$, $\lambda(\{c, v\}) \in L'$. To this end, we show that for each $i \in [1, \Delta - 3]$, the edge $\{v_i^*, c\}$ is the only edge incident with c of label $\lambda(\{v_i^*, c\})$. This then implies that $\lambda(\{v_i^*, c\}) \notin L'$ for each $i \in [1, \Delta - 3]$, which then results into L' having size at most 3. Let w be an arbitrary vertex of $(V^* \cup V) \setminus \{v_i^*\}$. Since v_i^* and w have distance exactly 2, there is a path of duration less than $\Delta+1$ by the fact that $\alpha \cdot D_{v_i^*,w} < 2 \cdot D_{v_i^*,w} = 2 \cdot \frac{\Delta+1}{2} = \Delta+1$. By construction, $P := (v_i^*, c, w)$ is the only path from v_i^* to w of length (and thus duration) less than $\Delta + 1 > \alpha \cdot D_{v_i^*,w}$. For P to have duration less than $\Delta + 1$, the edges $\{v_i^*, c\}$ and $\{w, c\}$ receive distinct labels under λ. Since vertex w was chosen arbitrarily, this implies that edge $\{v_i^*, c\}$ is the only edge incident with c of label $\lambda(\{v_i^*, c\})$. This further concludes that L' has size at most 3.

We now define a 3-coloring $\chi\colon V \to L'$ as follows: For each vertex $v \in V$, we set $\chi(v) := \lambda(\{c, v\})$. Next, we show that for each edge $\{u, v\} \in E$, u and v receive distinct colors under χ. Since $\{u, v\}$ is an edge of E, there is no sunglasses gadget with docking points u and v in G'. This implies that (u, c, v) is the only path from u to v in G' of length less than $\Delta + 1 > \alpha \cdot D_{u,v}$. Assume towards a contradiction that $\chi(u) = \chi(v)$. This would imply that $\lambda(\{c, u\}) = \lambda(\{c, v\})$. Hence, the unique path (u, c, v) from u to v in G' of length less than $\Delta + 1$ has a duration of exactly $\Delta + 1 > 2 \cdot \alpha = \alpha \cdot D_{u,v}$. This contradicts the assumption that λ is an edge labeling of G' with stretch at most α. Thus, $\chi(u) = \lambda(\{c, u\}) \neq \lambda(\{c, v\}) = \chi(v)$, which implies that χ is a proper 3-coloring for G'. □

It remains to show the backwards direction. To this end, suppose that G is 3-colorable. Let $\chi\colon V \to \{1, 2, 3\}$ be a 3-coloring of G. Assume without loss of generality that each of the three colors is assigned at least once.

The labeling. We define an edge labeling λ of G' of stretch $\frac{\Delta}{2}$ as follows: Let

$$\ell_2 := \begin{cases} \frac{\Delta}{2} & \Delta \text{ is even, and} \\ \frac{\Delta+1}{2} & \text{otherwise.} \end{cases}$$

Moreover, let $L := \{\ell_1 := \ell_2 - 1, \ell_2, \ell_3 := \ell_2 + 1\}$. For each $i \in [1, \ell_1 - 1]$, we set $\lambda(\{c, v_i^*\}) := i$ and for each $i \in [\ell_3 + 1, \Delta]$, we set $\lambda(\{c, v_{i-3}^*\}) := i$. For each vertex $v \in V$, we set $\lambda(\{c, v\}) := \ell_{\chi}(v)$. For each non-edge $\{a, b\}$ of G (that is, for each added sunglasses gadget), we label the edges of the sunglasses gadget $S_{x,y}$ according the sunglasses labeling. To complete the definition of λ, we distinguish between three cases:

- For even $\Delta > 4$, it remains to define the labels of the edges between the vertices of X. For each of these edges e, we set $\lambda(e) := \ell_2$.

- For $\Delta = 4$, it remains to define the labels of the edges between the vertices of X and the labels of the edges incident with the vertices c_1, c_2, and c_3. Let $i \in \{1,2,3\}$. Recall that $N_{G'}[c_i] = \{c\} \cup X \cup \{c_1, c_2, c_3\}$. We set $\lambda(\{x, c_i\}) := i$ and for each vertex $x \in X$, we set $\lambda(\{c_i, x\}) := \begin{cases} 2 & i = 3, \text{ and} \\ 4 & \text{otherwise.} \end{cases}$

Additionally, we set $\lambda(\{c_1, c_2\}) := 3$, $\lambda(\{c_1, c_3\}) := 2$, and $\lambda(\{c_2, c_3\}) := 1$.

- For odd Δ, it remains to define the labels of the edges incident with the vertices of $\widehat{X}$. For each vertex $\widehat{x} \in \widehat{X}$, we set $\lambda(\{x, \widehat{x}\}) := \ell_1$. For each other edge e incident with at least one vertex of $\widehat{X}$, we set $\lambda(e) := \ell_2$.

This completes the definition of λ. Next, we show that λ has a stretch of $\frac{\Delta}{2} \leq \alpha$. To this end, we first observe the existence of temporal paths of duration at most $3 \cdot \frac{\Delta}{2}$ between the vertices of $V \cup Y \cup X \cup \widehat{X}$.[9] Afterwards, between the vertices of $V \cup Y \cup X \cup \widehat{X}$, we then only need to consider vertex pairs of distance 2. Additionally, we consider the vertex pairs that include (at least) one vertex of $\{c\} \cup V^*$ in the process. The total proof is rather long because we have to distinguish between even and odd values of Δ, as well as distinguishing between $\Delta = 4$ and $\Delta > 4$. The arguments are mostly similar, but some individual parts of the proof cannot be unified. In the following, we thus only formally prove the cases for odd $\Delta > 4$. The formal proofs for the remaining cases are given in the appendix.

Lemma 20. *Let v and w be distinct vertices of $V \cup Y \cup X \cup \widehat{X}$. There is a path of duration at most $3 \cdot \frac{\Delta}{2}$ from v to w. If $v \in V$, then (i) the first edge of this path has label Δ or (ii) this path has duration less than Δ and the first edge of this path has label 1. If $w \in V$, then the last edge of this path has label Δ. Moreover, if any of v or w is from $X \cup \widehat{X}$, this path has length at most Δ.*

Proof. To show Lemma 20 we distinguish between even and odd Δ. First, we consider the even case.

Claim 20.1 ($\star$)**.** *Lemma 20 holds if Δ is even.*

Next, we consider the odd case.

Claim 20.2. *Lemma 20 holds if Δ is odd.*

[9] Recall that $\widehat{X} = \emptyset$ for even Δ.

Proof of Claim. To prove the statement, we show the existence of two special kinds of fast paths:

- For each $\widehat{x} \in \widehat{X}$ and each $v \in V \cup Y \cup X$ where x and v are not part of a common chronological cycle, there is a path $P_{\widehat{x},v}$ of duration at most $\Delta - 1$ from $\widehat{x}$ to v that starts with an edge of label ℓ_2.

- For each $v \in V \cup Y$, there is a path P_v of duration at most ℓ_1 from v to some vertex $x \in X$ which is part of a common chronological cycle with v, such that P_v ends with an edge of label $\ell_1 - 1$.

We now prove the existence of the first type of fast paths. Let $\widehat{x} \in \widehat{X}$ and let $v \in V \cup Y \cup X \cup \widehat{X}$.

Case: v is part of a common chronological cycle C with x**.** If v and $\widehat{x}$ are adjacent, the path $P_{\widehat{x},v} := (\widehat{x}, v)$ has duration $1 < \Delta$. Otherwise, that is, if v and x are not adjacent, consider the path $P_{\widehat{x},v}$ that first traverses the edge from $\widehat{x}$ to x at time ℓ_1 and then follows C until reaching vertex v. Since v is not adjacent to $\widehat{x}$ (that is, v is not a vertex of X), this path has duration at most $\Delta < 3 \cdot \frac{\Delta}{2}$. In particular, if v is a vertex of V, the last edge of this path has label Δ.

Case: v is not part of a common chronological cycle C with x**.** The sunglasses labeling thus implies that there is a vertex $z \in X$, such that z is part of a common chronological cycle C with v, and the edge of z to its successor in C has label ℓ_3. If v is the predecessor of z in C, then v is a vertex of X and the edge $\{\widehat{x}, v\}$ exists and has label ℓ_2. Hence, $P_{\widehat{x},v} := (\widehat{x}, v)$ has duration $1 < \Delta - 1$ and starts at time ℓ_2. Otherwise, that is, if v is not the predecessor of z in C, consider the path $P_{\widehat{x},v}$ that first traverses the edge from $\widehat{x}$ to z at time ℓ_2 and then follows C until reaching vertex v. This path has a duration of at most $\Delta - 1$ and starts with label ℓ_2. Moreover, if v is a vertex of V, the last edge of this path has label Δ. This proves the existence of the first type of fast paths.

We now prove the existence of the second type of fast paths. Let $v \in V \cup Y$. If $v \in Y$, by definition of the sunglasses labeling, there is a zigzag path Z from v to some vertex $x \in X$ which is part of the same chronological cycle, such that Z has a duration less than ℓ_1 and reaches vertex x at time $\ell_1 - 1$. If $v \in V$, then there is a chronological cycle C attached to v, and v has an incident edge in C of label Δ. Let s denote the endpoint of this edge. For $\Delta > 3$, the above discussed zigzag path Z from s to some vertex of X starts with label 1. Hence, by going from v to s at time Δ and then following the zigzag path from s to x, we obtain a desired path from v to some vertex of X. Moreover, this path starts with label Δ and has a duration of at most ℓ_1 due to the duration of the zigzag path Z. This shows the existence of the second type of fast paths.

Since the edges between the vertices of X and $\widehat{X}$ all receive labels from $\{\ell_1, \ell_2\}$ the paths P_v can be extended to obtain paths from each vertex of $V \cup Y$ to each vertex of $\widehat{X}$ with duration at most $\ell_1 + 2 < \frac{\Delta}{2} + 2 < 3 \cdot \frac{\Delta}{2}$. This is due to the fact that P_v reaches some vertex of X with an edge of label $\ell_1 - 1$ with a duration of at most ℓ_1.

Hence, between any vertex $\widehat{x} \in \widehat{X}$ and any vertex $v \in V \cup Y \cup X$, there are paths of duration at most $3 \cdot \frac{\Delta}{2}$ each, since the described path $P_{\widehat{x},c}$ has a duration of at most Δ. It thus remains to consider fast paths between vertices v and w of $V \cup X \cup Y$.

Case: v and w are part of a common chronological cycle C**.** Hence, there are paths of duration at most $\Delta - 1$ between v and w each, by following their common chronological cycle. In particular, if $v \in V$, the first edge of the path from v to w has label 1 and the last edge of the path from w to v has label Δ. Hence, the statement holds for pairs of vertices that are part of some common chronological cycle.

Case: v and w are not part of any common chronological cycle**.** Due to symmetry, we only describe a fast path from v to w. By the above, there is some path P_v of duration at most ℓ_1 that reaches some vertex $x \in X$ with an edge of label $\ell_1 - 1$. Moreover, this vertex x is part of a common chronological cycle with v. Since v and w are not part of any common chronological cycle, x is not part of a common chronological cycle with w. Hence, the described path $P_{\widehat{x},w}$ from $\widehat{x}$ to w has duration at most Δ and starts with an edge of label ℓ_2. Now, consider the path $P_{v,w}$ obtained by first following the path P_v to vertex x, afterwards going to $\widehat{x}$ with label ℓ_1, and then following the path $P_{\widehat{x},w}$ starting with label $\ell_2 = \ell_1 + 1$. The duration of this path $P_{v,w}$ is at most $d(P_v) + 1 + d(P_{\widehat{x},w}) \leq \ell_1 + 1 + \Delta - 1 = \frac{\Delta - 1}{2} + \Delta < 3 \cdot \frac{\Delta}{2}$, where d denotes the duration of the respective path. Moreover, if v is a vertex of V, then $P_{v,w}$ starts with label Δ, since P_v starts with label Δ. Similarly, if w is a vertex of V, then $P_{v,w}$ ends with label Δ, since P_w ends with label Δ. ■

Now, Lemma 20 follows from Claim 20.1 and Claim 20.2. □

Recall that Lemma 20 states that for each vertex $v \in V \cup Y \cup X \cup \widehat{X}$ and each vertex $w \in V \cup Y \cup X \cup \widehat{X}$ there are paths of duration at most $3 \cdot \frac{\Delta}{2}$ each between v to w. Hence, if v and w have distance at least 3, then these paths have a duration of at most $\frac{\Delta}{2} \cdot D_{a,b} \leq \alpha \cdot D_{a,b}$. Moreover, the last statement of Lemma 20 states that whenever v or w is from $X \cup \widehat{X}$, then the duration of these paths is at most $\Delta \leq \frac{\Delta}{2} \cdot D_{a,b} \leq \alpha \cdot D_{a,b}$. To show that the stretch of λ is at most $\frac{\Delta}{2}$, it thus remains to consider (i) the durations of fastest paths from and to vertices of $\{c\} \cup V^*$, (ii) the durations of fastest paths between vertex pairs from $Y \cup V$ of distance exactly two, and (iii) if $\Delta = 4$, fastest paths from and to vertices of $\{c_1, c_2, c_3\}$.

First, we consider the duration of fastest paths from and to vertices of $\{c\} \cup V^*$. To this end, we distinguish between the different values of Δ. We distinguish between the cases of $\Delta = 4$ and $\Delta > 4$.

First, we consider $\Delta = 4$.

Lemma 21 ($\star$). *Let $\Delta = 4$, let q be a vertex of $V^* \cup \{c, c_1, c_2, c_3\}$, and let z be a vertex of V'. There is a path from q to z and a path from z to q that each have a duration of at most $\frac{\Delta}{2} \cdot D_{q,z} = 2 \cdot D_{q,z}$.*

Next, we consider $\Delta > 4$.

Lemma 22. *Let $\Delta > 4$, let q be a vertex of $V^* \cup \{c\}$, and let z be a vertex of V'. Then, there is a path from q to z and a path from z to q that each have a duration of at most $\frac{\Delta}{2} \cdot D_{q,z}$.*

Proof. Recall that $L := \{\ell_1, \ell_2, \ell_3\}$ are the labels of the edges between c and the vertices of V. Moreover note that $L \cap \{1, \Delta\} = \emptyset$ since $\Delta > 4$.

Firstly, we consider the durations from and to the center vertex c. Recall that for each vertex $v \in V$, $\lambda(\{v, c\}) \in L$.

Let z be a vertex of distance exactly 2 with c. By construction, there is a vertex $v \in V$ which is a common neighbor of c and z. Due to the sunglasses labeling, edge $\{v, z\}$ receives label either 1 or Δ. By definition of λ, edge $\{c, v\}$ receives a label from L. Since $\Delta > 4$, $L \cap \{1, \Delta\} = \emptyset$. Hence, v is a nice neighbor of c and z.

Let z be a vertex of distance exactly 3 with c. Since $\Delta > 3$, z is part of a chronological cycle $C_{\overline{e}}^{v}$ for some none-edge $\overline{e}$ of G and some vertex $v \in \overline{e}$. In particular, $z \in \{s_{\overline{e}}^{v,2}, s_{\overline{e}}^{v,\Delta-2}\}$. Let $\ell' := \lambda(\{c, v\})$. Recall that $\ell \in L$ and that $\{s_{\overline{e}}^{v,2}, s_{\overline{e}}^{v,\Delta-2}\}$ is a parallel edge with label $\frac{\Delta}{2}$. Consider the path $P_{c,z} := (c, v, s_{\overline{e}}^{v,1}, s_{\overline{e}}^{v,2}, s_{\overline{e}}^{v,\Delta-2})$. This path uses edges with labels ℓ', 1, 2, and $\frac{\Delta}{2}$. Hence, $P_{c,z}$ has a duration of $\Delta + \frac{\Delta}{2} - \ell' + 1 \leq \Delta + \frac{\Delta}{2} - \ell_1 + 1 = \Delta + 2$. Since $P_{c,z}$ contains vertex z, there is a path of duration at most $\Delta + 2 \leq 3 \cdot \frac{\Delta}{2} \leq \frac{\Delta}{2} \cdot D_{c,z} \leq \alpha \cdot D_{c,z}$. Now, consider the path $P_{z,c} := (s_{\overline{e}}^{v,2}, s_{\overline{e}}^{v,\Delta-2}, s_{\overline{e}}^{v,\Delta-1}, v, c)$.

This path uses edges with labels $\frac{\Delta}{2}$, $\Delta - 1$, Δ, and ℓ'. Hence, $P_{z,c}$ has a duration of $\Delta + \ell' - \frac{\Delta}{2} + 1 \leq \Delta + \ell_3 - \frac{\Delta}{2} + 1 = \Delta + 2$. Since $P_{z,c}$ contains vertex z, there is a path of duration at most $\Delta + 2 \leq 3 \cdot \frac{\Delta}{2} \leq \frac{\Delta}{2} \cdot D_{z,c} \leq \alpha \cdot D_{z,c}$.

Let z be a vertex of distance at least 4 with c.[10] Moreover, for $i \in \{1, 3\}$ let v_i denote some vertex of V with $\chi(v_i) = i$, that is, a vertex with $\lambda(\{v, c\}) = \ell_i$. By assumption, such vertices exist. Moreover, recall that due to Lemma 20, there is a path P_3 from v_3 to z of (a) duration at most $3 \cdot \frac{\Delta}{2}$ that starts by traversing an edge with label Δ or (b) duration at most $\Delta \leq 3 \cdot \frac{\Delta}{2} - 1$ that starts by traversing an edge with label 1. Similarly, due to Lemma 20, there is a path P_1 from z to v_1 of duration at most $3 \cdot \frac{\Delta}{2}$ that ends by traversing an edge with label Δ. Consider the path $P_{c,z}$ starting at c, moving to v_3, and afterwards following the path P_3. Since P_3 has a duration of at most $3 \cdot \frac{\Delta}{2}$, $P_{c,z}$ reaches z at time at most $\Delta + 3 \cdot \frac{\Delta}{2} - 1$. Hence, the duration of $P_{c,z}$ is at most $\Delta + 3 \cdot \frac{\Delta}{2} - 1 - \lambda(\{c, v_3\}) + 1 = \Delta + 3 \cdot \frac{\Delta}{2} - \ell_3 \leq 2 \cdot \Delta$. Consequently, $P_{c,z}$ has a duration of at most $2 \cdot \Delta = 4 \cdot \frac{\Delta}{2} = \frac{\Delta}{2} \cdot D_{c,z} \leq \alpha \cdot D_{c,z}$. For the opposite direction, consider path $P_{z,c}$ that starts at z, follows P_1 to vertex v_1, and then moves to c. Since P_1 has a duration of at most $3 \cdot \frac{\Delta}{2}$, $P_{z,c}$ starts at the latest at time $\Delta - \frac{\Delta}{2} + 1 \geq \ell_3$ and reaches c at time ℓ_1. Hence, the duration of $P_{z,c}$ is at most $2 \cdot \Delta + \ell_1 - \ell_3 + 1 = 2 \cdot \frac{\Delta}{2} - 1 \leq 2 \cdot \Delta - \frac{1}{2}$. Consequently, $P_{z,c}$ has a duration of at most $2 \cdot \Delta = 4 \cdot \frac{\Delta}{2} = \frac{\Delta}{2} \cdot D_{z,c} \leq \alpha \cdot D_{z,c}$.

Hence, the statement holds for c.

Secondly, we consider the duration of paths from and to vertices of V^*. Let v_i^* be a vertex of V^* and let $\ell^* := \lambda(\{v_i^*, c\})$. Recall that c is the only neighbor of v_i^* and that $\ell^* \notin L$.

Let z be a vertex of distance exactly 2 with c. By construction, z is a vertex of V and edge $\{v_i^*, c\}$ is the only edge incident with c of label $\lambda(\{v_i^*, c\})$. This implies that c is a nice neighbor of v_i^* and z.

Let z be a vertex of distance exactly 3 with v_i^*. By construction, $z \in \{s_{\overline{e}}^{v,1}, s_{\overline{e}}^{v,\Delta-1}\}$ for some none-edge $\overline{e}$ of G and some vertex $v \in \overline{e}$. Let $\ell_v := \lambda(\{c, v\})$ and let $\ell_z := \lambda(\{v, z\})$. Recall that $\ell_v \in L$ and that $\ell_z \in \{1, \Delta\}$. Consider the paths $P_{v_i^*,z} := (v_i^*, c, v, z)$ and $P_{z,v_i^*} := (z, v, c, v_i^*)$. These paths uses edges with labels ℓ^*, ℓ_v, and ℓ_z from left to right or right to left. If $\ell^* < \ell_1$, then the duration of $P_{v_i^*,z}$ is maximized when $\ell^* = \ell_z = 1$, in which case the duration of $P_{v_i^*,z}$ is $\Delta + 1 \leq 3 \cdot \frac{\Delta}{2}$. Moreover, the duration of P_{z,v_i^*} is maximized when $\ell^* = \ell_1 - 1$ and $\ell_z = \Delta$, in which case the duration of P_{z,v_i^*}

[10] Note that for odd Δ this includes all vertices of $\widehat{X}$, since $\Delta \geq 5$ in this case.

is $2 \cdot \Delta + \ell^* - \ell_z + 1 = \Delta + \ell_1 < 3 \cdot \frac{\Delta}{2} \leq \frac{\Delta}{2} \cdot D_{z,v_i^*} \leq \alpha \cdot D_{z,v_i^*}$. Otherwise, that is, if $\ell^* > \ell_3$, then the duration of $P_{v_i^*,z}$ is maximized when $\ell^* = \ell_3 + 1$ and $\ell_z = 1$, in which case the duration of $P_{v_i^*,z}$ is $2 \cdot \Delta + \ell_z - \ell^* + 1 = \Delta - \ell_3 + 1 \leq \Delta - (\ell_1 + 2) + 1 \leq \Delta - \frac{\Delta}{2} = 3 \cdot \frac{\Delta}{2}$. Moreover, the duration of P_{z,v_i^*} is maximized when $\ell^* = \ell_z = \Delta$, in which case the duration of P_{z,v_i^*} is $\Delta + 1 < 3 \cdot \frac{\Delta}{2}$. In both cases, we presented paths between v_i^* and z of duration at most $3 \cdot \frac{\Delta}{2} \leq \frac{\Delta}{2} \cdot D_{z,v_i^*} \leq \alpha \cdot D_{z,v_i^*}$.

Let z be a vertex of distance exactly 4 with c_i^*. Hence, z has distance 3 to c. Consider the two paths $P_{c,z}$ and $P_{z,c}$ described above for this case: there is a vertex $v \in V$, such that (i) $P_{c,z}$ contained the vertex z, started by using the edge $\{c, v\}$, and ended in time step $\Delta + \ell_2$, and (ii) $P_{z,c}$ contained the vertex z, started at time step ℓ_2, and ended by traversing edge $\{c, v\}$ at time step $\Delta + \lambda(\{c, v\})$. Let $\ell_v := \lambda(\{c, v\})$. Now consider the path $P_{v_i^*,z}$ obtained from first moving from v_i^* to c and then following the path $P_{c,z}$ and the path P_{z,v_i^*} obtained from first following the path $P_{z,c}$ to vertex c and then traversing the edge towards v_i^*. If $\ell^* < \ell_1$, then $\ell^* < \ell_v$, which implies that $P_{v_i^*,z}$ reaches vertex z at times step at most $\Delta + \ell_2$. Hence, $P_{v_i^*,z}$ has a duration of at most $\Delta + \ell_2 - \ell^* + 1 \leq 2 \cdot \Delta$. Moreover, the path P_{z,v_i^*} reaches vertex c at time step $\Delta + \ell_v$ and can thus reach vertex v_i^* at time step $2 \cdot \Delta + \ell^*$. Hence, P_{z,v_i^*} has a duration of at most $2 \cdot \Delta + \ell^* - \ell_2 + 1 \leq 2 \cdot \Delta + \ell_1 - 1 - \ell_2 + 1 < 2 \cdot \Delta$. Otherwise, that is, if $\ell^* > \ell_3$, then $\ell^* > \ell_v$, which implies that $P_{v_i^*,z}$ reaches vertex z at times step at most $2 \cdot \Delta + \ell_2$. Hence, $P_{v_i^*,z}$ has a duration of at most $2 \cdot \Delta + \ell_2 - (\ell_3 + 1) + 1 < 2 \cdot \Delta$. Moreover, the path P_{z,v_i^*} reaches vertex c at time step $\Delta + \ell_v$ and can thus reach vertex v_i^* at time step $\Delta + \ell^*$. Hence, P_{z,v_i^*} has a duration of at most $\Delta + \ell^* - \ell_2 + 1 \leq 2 \cdot \Delta$. In both cases, we presented paths between v_i^* and z each having a duration of at most $2 \cdot \Delta = 4 \cdot \frac{\Delta}{2} = \frac{\Delta}{2} \cdot D_{v_i^*,z}$.

Let z be a vertex of distance at least 5 with v_i^*. Hence, there is a sunglasses gadget $S_{\{v,w\}}$ that contains z.[11] Without loss of generality assume that z is contained in the chronological (v, w)-path P^v of $S_{\{v,w\}}$. Recall that the labeling of this path P^v uses the labels of $[1, \Delta]$ consecutively. Hence, there is path $P_{v,z}$ of duration at most $\Delta - 1$ from v to z that starts with label 1, and there is a path $P_{z,w}$ of duration at most $\Delta - 1$ from z that reaches w in time step Δ. Consider the path $P_{v_i^*,z}$ that starts at v_i^*, moves to c, moves to v, and then follows the path $P_{v,z}$. The duration of this path is maximized,

[11] Note that for odd Δ this includes all vertices of $\widehat{X}$, since $\Delta \geq 5$ in this case.

if $\ell^* = \ell_3+1$, in which case vertex z is reached at time step $2\cdot\Delta+\Delta-1$. Hence, the duration of $P_{v_i^*,z}$ is at most $2\cdot\Delta+\Delta-1-\ell^*+1 = 3\cdot\Delta-(\ell_1+3)+1 \leq 5\cdot\frac{\Delta}{2}$. Finally, consider the path P_{z,v_i^*} that starts at z, follows $P_{z,w}$ to vertex w, moves to c, moves to v_i^*. The duration of this path is maximized, if $\ell^* = \ell_1-1$, in which case vertex z is reached at time step $2\cdot\Delta+\ell^*-1$. Hence, the duration of P_{z,v_i^*} is at most $2\cdot\Delta+\ell^*-1+1 = 2\cdot\Delta+\ell_1-1 \leq 5\cdot\frac{\Delta}{2}$. Both paths thus have a duration of at most $5\cdot\frac{\Delta}{2} \leq \frac{\Delta}{2}\cdot D_{v_i^*,z}$. □

Hence, in all cases for Δ, for each vertex $q \in V' \setminus (V \cup Y \cup X \cup \widehat{X})$ and each vertex $z \in V'$, there are paths between q and z of durations at most $\frac{\Delta}{2}\cdot D_{q,z}$ each.

It thus remains to consider the durations of fastest paths between vertex pairs (a,b) from $Y \cup V$ of distance exactly two. First, we will analyze the vertex pairs containing exactly one vertex of V and exactly one vertex of Y. Let $v \in V$ and let $y \in Y$, such that v and y have distance exactly 2, then by construction, y is a vertex of a chronological cycle attached to v for some sunglasses gadget. Hence, by taking the fastest paths along the chronological cycle, both vertices can pairwise reach each other with a path of duration less than $\Delta = \frac{\Delta}{2}\cdot D_{u,v} \leq \alpha\cdot D_{u,v}$. Now, consider the vertex pair of Y of distance exactly 2.

Lemma 23. *Let y and y' be vertices of Y of distance exactly 2. Then, there is a path of duration at most Δ from y to y'.*

Proof. Recall that the statement holds, if there is a nice neighbor of y and y'. By construction, the only vertices of Y of distance exactly 2 are (i) contained in the same sunglasses gadget or (ii) are both neighbors of some vertex $v \in V$ and contained in distinct chronological cycles attached to v.

Firstly, consider the case where y and y' are contained in the same sunglasses gadget. If y and y' are part of the same chronological path of that sunglasses, then they only have one common neighbor which is a nice neighbor due to the sunglasses labeling. If y and y' are part of the same chronological cycle of that sunglasses gadget, then they can pairwise reach each other in duration at most Δ each by following the chronological cycle. For even Δ, we also need to consider the case that y and y' are neither part of the same chronological path nor part of the same chronological cycle. For odd Δ, this case cannot occur, since vertices of Y of two disjoint chronological cycles of the same sunglasses gadget are separated by two vertices of X. If y and y' are neither part of the same chronological path nor part of the same chronological cycle, then by definition of sunglasses gadgets, there is some central vertex $x \in X$ that is adjacent to both y and y', such that x is part of the chronological path that contains y but x is not part of the chronological path that contains y'. By definition of the sunglasses labeling, the edge $\{x, y\}$ receives a label from $\{\ell_2, \ell_3\}$ under λ and the edge $\{x, y'\}$ is part of a zigzag path and receives the label ℓ_1 under λ. Hence, x is a nice neighbor of y and y'.

Secondly, consider the case where y and y' are not contained in the same sunglasses gadget. Let v be the unique common neighbor of y and y' in V and let C (C') be the chronological cycle attached to v that contains y (y'). By definition of the sunglasses labeling, $\{\lambda(\{y, v\}), \lambda(\{y', v\})\} \subseteq \{1, \Delta\}$. If v is a nice neighbor of y and y', the statement is shown. So, consider $\lambda(\{y, v\}) = \lambda(\{y', v\})$. Due to symmetry, we only describe a fast path from y to y'. We distinguish two cases. If $\lambda(\{y, v\}) = \lambda(\{y', v\}) = 1$, consider the path P that starts at y, goes through the whole chronological cycle C to v with label Δ, and finally traverses the edge towards y' with label 1. This path has a duration of at most Δ, since the edge of smallest label of C that P uses is at least 2, since $\{y, v\}$ is the only edge of C with label 1. Otherwise, that is, if $\lambda(\{y, v\}) = \lambda(\{y', v\}) = 1$, we do essentially the same. Consider the path P that starts at y, goes over the direct edge to v with label Δ, and goes through the whole chronological cycle C' until reaching y' with a label of at most $\Delta - 1$. Hence, P has duration at most Δ. □

It remains to show that between any two vertices u and v of V, there is a path of duration at most $\Delta = \frac{\Delta}{2} \cdot D_{u,v} \leq \alpha \cdot D_{u,v}$. If $\{u, v\}$ is a non-edge of G, then there is a sunglasses gadget with docking points u and v in G'. By following the chronological (u, v)-path ((v, u)-path) of this sunglasses gadget, there is a temporal path from u (v) to v (u) of duration Δ. Otherwise, that is, if $\{u, v\}$ is an edge of E, then (u, c, v) and (v, c, u) are both paths of duration at most Δ each, since c is a nice neighbor of u and v. This is due to the fact that χ is a proper coloring of G, which implies that $\chi(u) \neq \chi(v)$ and thus $\lambda(\{u, c\}) = \ell_{\chi}(u) \neq \ell_{\chi}(v) = \lambda(\{v, c\})$. Concluding, for each pair (a, b) of vertices of V', there is a path of duration at most $\frac{\Delta}{2} \cdot D_{a,b}$, which implies that λ has a stretch of at most $\frac{\Delta}{2} \leq \alpha$.

With all of this, we conclude that the stretch of λ is at most $\frac{\Delta}{2}$. This implies that G is 3-colorable only if I is a yes-instance of SPTGR. Together with Lemma 19, this now completes the proof of Theorem 18.

Proof of Theorem 18. Based on the above construction and arguments, we conclude Theorem 18: If G admits a proper 3-coloring χ, then we showed above that the labeling λ for G' provides a stretch of at most $\frac{\Delta}{2} \leq \alpha$. If there is labeling λ for G' of stretch at most α, then Lemma 19 implies that G is 3-colorable. Consequently, the reduction is correct and proves the stated hardness results for SPTGR. □

Recall that Lemma 10 shows that SPTGR is NP-hard for each $\alpha \in [1, 1.5)$. Moreover, since for each constant $\alpha \geq 1.5$, there is a constant $\Delta \geq 3$, such that $\alpha \in [\frac{\Delta}{2}, \frac{\Delta+1}{2})$, Lemmas 10 and 14 and theorem 18 imply the following.

Theorem 24. *For each constant* $\Delta \geq 3$*,* SPTGR *is NP-hard. For each constant* $\alpha \geq 1$*,* SPTGR *is NP-hard.*

6. Fixed-parameter Algorithms through Monadic Second-Order Logic

In this section, we present monadic second-order (MSO) logic [9, 10] formulations for SPTGR. Expressing problems in MSO is a powerful tool that together with Courcelle's famous theorem (and extension thereof) [9, 10] yields fixed-parameter tractability for certain parameters depending on the formula size and the extesions of MSO that are used. Informally speaking, we aim to provide the smallest formulations that need the least amount of extensions to MSO. In particular, we show that SPTGR is fixed-parameter tractable with respect to combinations of Δ, the treewidth $\text{tw}(G)$, the diameter $\text{diam}(G)$, and the neighborhood diversity $\text{nd}(G)$ of the input graph G. Formally, we show the following two results.

Theorem 25. SPTGR *is in FPT when parameterized by* $\text{nd}(G) + \Delta$*.*

Theorem 26. SPTGR *is in FPT when parameterized by* $\text{tw}(G)+\text{diam}(G)+\Delta$*.*

We remark that our hardness results (in particular, Theorem 18) show that fixed-parameter tractability presumably cannot be obtained with only $\text{diam}(G) + \Delta$ as parameter. Hence, we need a parameter that is larger or incomparable, as we have in Theorems 25 and 26. However, we leave open whether the parameters $\text{tw}(G)$ and $\text{nd}(G)$ by themselves can already yield fixed-parameter tractability.

Furthermore, note that Theorem 26 implies that for all graph classes that have locally bounded treewidth (such as planar graphs), we get that SPTGR is in FPT when parameterized by the diameter of the input graph and Δ. Furthermore, Theorem 26 implies that SPTGR is in FPT when parameterized by the treedepth of the input graph and Δ, since both the treewidth and the diameter of a graph can be upper bounded by a function of the treedepth. We remark that the neighborhood diversity of a graph is unrelated to the combination of the treewidth and the diameter of a graph. Finally, by Lemma 3, both theorem statements also hold for the optimization version of SPTGR.

To show Theorems 25 and 26 we show that SPTGR is expressible in monadic second-order (MSO) logic in certain specific ways that allow us to employ Courcelle's famous theorem (and extensions thereof) [9, 10]. Since we define the MSO formulas directly on the input graph G, we do not give the formal definitions of treewidth and neighborhood diversity, since they are not necessary to obtain the results. For more information on treewidth, we refer to standard textbooks on graph theory, e.g. the one by Diestel [15], and for more information on the neighborhood diversity, we refer to Lampis [35], who introduced this parameter.

Assume we are given a graph $G = (V, E)$, an integer Δ, and a real number $\alpha \geq 1$. Let $n = |V|$. A *monadic second-order (MSO) formula* ϕ over G is a formula that uses

- the incidence relation of vertices and edges,
- the logical operators $\wedge$, $\vee$, $\neg$, $=$, and parentheses,
- a finite set of variables, each of which is either taken as an element or a subset of V or E,
- and the quantifiers $\forall$ and $\exists$.

Additionally we will use some folklore shortcuts such as $\Rightarrow$, $\neq$, $\subseteq$, $\in$, and $\setminus$, which can themselves be replaced by MSO formulas. For an edge set E' we use $V(E')$ to denote the set of vertices that are incident with the edges in E'. Furthermore, we use the following formula to express that sets $X_1, \ldots, X_i$

form a partition of X.

$$\text{partition}_i(X_1, X_2, \ldots, X_i, X) := \Big(\bigwedge_{1 \le i' \le i} X_{i'} \subseteq X\Big) \wedge \Big(\forall x \in X : \bigvee_{1 \le i' \le i} x \in X_{i'}\Big) \wedge \Big(\forall x \in X : \bigwedge_{1 \le i' < i'' \le i} (x \in X_{i'} \Rightarrow x \notin X_{i''})\Big)$$

Note that the formula $\text{partition}_i(X_1, X_2, \ldots, X_i, X)$ has size $\mathcal{O}(i^2)$.

We use two different extensions of MSO. The first one is called CMSO and allows for testing the cardinality of a set. We remark that this does not allow for comparing the cardinalities of sets.

- $\text{card}_{n,p}(X) = \text{true}$ if and only if $|X| \equiv n \mod p$.

This extension of MSO was already considered by Courcelle [9]. We have the following, where $|\phi|$ denotes the length of the formula ϕ.

Theorem 27 ([9, 10])**.** *CMSO model checking is in FPT when parameterized by* $\text{tw}(G) + |\phi|$.

The second extension is stronger and allows for linear cardinality constraints, that is, expressions of the type $x_1 \le x_2$, where x_1 and x_2 are linear expressions over cardinalities of sets. This extension is called $\text{MSO}^{\text{GL}}_{\text{lin}}$ [34] and the following is known.

Theorem 28 ([34])**.** *MSO^{GL}_{lin} model checking is in FPT when parameterized by* $\text{nd}(G) + |\phi|$.

We now introduce some basic MSO formulas that we will use to compose the formulas to express SPTGR. It is well-known that connectivity and various related concepts can be expressed in MSO.

- $\text{conngraph}(X, E')$ tests whether the subgraph $(X, E' \cap X^2)$ of G is connected:

  $$\text{conngraph}(X, E') := \forall \emptyset \ne Y \subset X \exists x \in X \setminus Y \exists y \in Y \exists e \in E' : x \in e \wedge y \in e$$

- $\text{conn}(v, w, E')$ tests whether the two vertices $v, w \in V$ are connected by a path that only uses edges from $E' \subseteq E$:

  $$\text{conn}(v, w, E') := \exists X \subseteq V : \text{conngraph}(X, E') \wedge v \in X \wedge w \in X$$

- $\mathrm{path}(v, w, E')$ tests whether the two vertices $v, w \in V$ are connected by a path that *exactly* uses edges from $E' \subseteq E$:

$$\mathrm{path}(v, w, E') := \mathrm{conn}(v, w, E') \wedge \forall E'' \subset E' : \neg\, \mathrm{conn}(v, w, E'')$$

Note that all the above-introduced formulas have constant size.

We first show Theorem 25. To this end, we introduce additional predicates that use linear cardinality constraints and hence are $\mathrm{MSO}^{\mathrm{GL}}_{\mathrm{lin}}$ formulas. Afterwards, we show how to replace these predicates with equivalent (larger) CMSO formulas.

- $\mathrm{spath}(v, w, E')$ tests whether the two vertices $v, w \in V$ are connected by a path that *exactly* uses edges from $E' \subseteq E$ and whether this is a shortest path:

$$\mathrm{spath}(v, w, E') := \mathrm{path}(v, w, E') \wedge \forall E'' : |E''| < |E'| \Rightarrow \neg\, \mathrm{conn}(v, w, E'')$$

Now we are ready to give an $\mathrm{MSO}^{\mathrm{GL}}_{\mathrm{lin}}$ formula $\phi_{G,\Delta,\alpha}$ that expresses SPTGR. We are looking for a partition of E into $E_1, E_2, \ldots, E_\Delta$. We interpret $e \in E_i$ with edge e receiving label i.

$$\phi_{G,\Delta,\alpha} = \exists E_1, \ldots, E_\Delta : \mathrm{partition}_\Delta(E_1, \ldots, E_\Delta, E) \wedge \tag{1}$$

$$\forall v, w\ \exists E^\star : \Big(\mathrm{path}(v, w, E^\star) \wedge \tag{2}$$

$$\exists X_1, \ldots, X_\Delta : \Big(\mathrm{partition}_\Delta(X_1, \ldots, X_\Delta, V(E^\star) \setminus \{v, w\}) \wedge \tag{3}$$

$$\bigwedge_{1 \le i \le \Delta} \Big(\forall x \in X_i\ \exists e_1, e_2 \in E^\star : \big(e_1 \neq e_2 \wedge x \in e_1 \cap e_2 \wedge \mathrm{conn}(v, x, E^\star \setminus \{e_1\}) \wedge \tag{4}$$

$$\bigvee_{1 \le i' \le \Delta} (e_1 \in E_{i'} \wedge e_2 \in E_{(i'+i) \bmod \Delta}) \big) \Big) \wedge \tag{5}$$

$$\exists E^{\star\star} : \mathrm{spath}(v, w, E^{\star\star}) \wedge \Big(\sum_{1 \le i \le \Delta} i \cdot |X_i| \Big) + 1 \le \alpha \cdot |E^{\star\star}| \Big) \Big) \tag{6}$$

Observe that the size of the formula is in $\mathcal{O}(\Delta^2)$ and that it can be computed in $\mathcal{O}(\Delta^2)$ time. Now we prove that ϕ expresses SPTGR.

Lemma 29. *Given an instance $I = (G, \Delta, \alpha)$ of* SPTGR*, we have that $\phi_{G,\Delta,\alpha}$ is satisfiable if and only if I is a yes-instance.*

Proof. Assume that $\phi_{G,\Delta,\alpha}$ is satisfiable. Then we label edge $e \in E$ with label i if and only if $e \in E_i$. Note that Line 1 of $\phi_{G,\Delta,\alpha}$ guarantees that $E_1, \ldots, E_\Delta$ is a partition of E and hence the labeling is well-defined. Now assume for contradiction that here is some $v, w \in V$ such that the duration of a fastest path from v to w is larger than $\alpha \cdot D_G(v, w)$. Consider Lines 2 to 6 have to hold for v, w. It follows from Line 2 that there exists an edge set $E^\star$ such that there is a path P from v to w in G that uses exactly the edges from $E^\star$. Line 3 implies that there is a partition $X_1, \ldots, X_\Delta$ of the vertices in $V(E^\star) \setminus \{v, w\}$, which are the internal vertices of P. Intuitively, a vertex $x \in X_i$ will imply that the waiting time at x is i. This is checked in the next two lines of the formula. For a vertex $x \in X_i$, Line 4 identifies the two edges e_1, e_2 that are incident with x and ensures that e_1 is used first. Line 5 ensures that the labels on e_1 and e_2 imply that the waiting time at x is indeed i. Finally, Line 6 identifies a shortest path from v to w in G and compares the duration of the temporal path along the edges of $E^\star$ with the distance between v and w (times α). In particular, it ensures that the duration of the temporal path along the edges $E^\star$ is at most $\alpha \cdot D_G(v, w)$. This is a contradiction to the assumption that the duration of a fastest path from v to w is larger than $\alpha \cdot D_G(v, w)$.

For the other direction, assume that there exists a labeling λ for G such that for all v, w the duration of a fastest path from v to w is at most $\alpha \cdot D_G(v, w)$. We argue that then, the formula $\phi_{G,\Delta,\alpha}$ is satisfiable. We define a partition $E_1, \ldots, E_\Delta$ of E as follows. For all $e \in E$ we put $e \in E_i$ if $\lambda(e) = i$. Since every edge receives exactly one label by λ this clearly forms a partition of E and hence the predicate in Line 1 of the formula evaluates to true. Consider Line 2 of the formula. For every v, w, we select $E^\star$ to be the edges of G that are visited by a fastest path P from v to u in the Δ-periodic temporal graph (G, λ). Clearly, the edges $E^\star$ form a path from v to w in G, and hence the predicate in Line 2 evaluates to true. Consider Line 3 next. We choose $X_1, \ldots, X_\Delta$ to be the following partition internal vertices of the path P. If the waiting time at vertex x of the temporal path P is i, we put $x \in X_{i+1}$. This clearly forms a partition and hence the predicate in Line 3 of the formula evaluates to true. Furthermore, by the construction of the partition, we get that Lines 4 and 5 evaluates to true. Finally, consider Line 6 of the formula. We choose $E^{\star\star}$ to be the edge set of a shortest path from v to w in G. Clearly, then the predicate evaluates to true, and the cardinality of $E^{\star\star}$ equals $D_G(v,w)$. The cardinality constraint is fulfilled since by assumption, the duration of P (which equals the sum of the waiting times at the internal vertices) is at most $\alpha \cdot D_G(v,w)$. We can conclude that $\phi_{G,\Delta,\alpha}$ is satisfiable. □

Now we have all the ingredients to prove Theorem 25.

Proof of Theorem 25. Theorem 25 follows directly from Lemma 29, Theorem 28, and from the fact that $\phi_{G,\Delta,\alpha}$ is an $\mathrm{MSO}^{\mathrm{GL}}_{\mathrm{lin}}$-formula with a size in $\mathcal{O}(\Delta^2)$ that can be computed in $\mathcal{O}(\Delta^2)$ time. □

To obtain Theorem 26, we replace all parts of the formula $\phi_{G,\Delta,\alpha}$ that use linear constrains over the cardinality of sets with equivalent ones that only use the predicate $\mathrm{card}_{n,p}$ from CMSO. This concerns mainly Line 6 of $\phi_{G,\Delta,\alpha}$. We show in the following that using the cardinality extension, we can test whether two vertices have a certain distance.

- $\mathrm{path}_{i,n+1}(v,w)$ tests whether the two vertices $v, w \in V$ are connected by a path of length i:

$$\mathrm{path}_{i,n+1}(v,w) := \exists E' : \mathrm{path}(v,w,E') \wedge \mathrm{card}_{i,n+1}(E')$$

- $\mathrm{dist}_{i,n+1}(v,w)$ tests whether the distance between the two vertices $v, w \in V$ is i:

$$\mathrm{dist}_{i,n+1}(v,w) := \mathrm{path}_{i,n+1}(v,w) \wedge \bigwedge_{1 \le i' < i} \neg\, \mathrm{path}_{i',n+1}(v,w)$$

We remark that the formula $\mathrm{dist}_{i,n+1}(v,w)$ has size $\mathcal{O}(i)$.

Next, we give a formula that, for some fixed d, checks whether the duration of a temporal path is exactly d. More specifically, the formula checks whether the sum of waiting times according to the partition $X_1, \ldots, X_\Delta$ from Line 3 equals d. To this end we need to consider all vectors $(\ell_1, \ldots, \ell_\Delta)$ such that if $|X_i| = \ell_i$ for all $1 \le i \le \Delta$, then $(\sum_i i \cdot |X_i|) + 1 = d$. Let $\mathcal{L}_d$ denote the set of all such vectors.

- $\mathrm{duration}_d(X_1, \ldots, X_\Delta)$ checks whether there exists a $(\ell_1, \ldots, \ell_\Delta) \in \mathcal{L}_d$ such that $|X_i| = \ell_i$ for all $1 \le i \le \Delta$:

$$\mathrm{duration}_d(X_1, \ldots, X_\Delta) := \bigvee_{(\ell_1,\ldots,\ell_\Delta) \in \mathcal{L}_d} \left(\bigwedge_{1 \le i \le \Delta} \mathrm{card}_{\ell_i,n+1}(X_i) \right)$$

To obtain the size of the above formula, we need to estimate the number of elements in $\mathcal{L}_d$. A straightforward bound is $|\mathcal{L}_d| \in \mathcal{O}(d^\Delta)$. Hence, the size of $\text{duration}_d(X_1, \ldots, X_\Delta)$ is in $\mathcal{O}(\Delta \cdot d^\Delta)$. Note that the set $\mathcal{L}_d$ (and hence the formula $\text{duration}_d(X_1, \ldots, X_\Delta)$) can also be computed in $\mathcal{O}(d^\Delta)$ time by enumerating all possible vectors $(\ell_1, \ldots, \ell_\Delta)$ and checking whether $(\sum_i i \cdot \ell_i) + 1 = d$.

Now we are ready to give a CMSO formula $\psi_{G,\Delta,\alpha}$ that expresses SPTGR.

$$\psi_{G,\Delta,\alpha} = \exists E_1, \ldots, E_\Delta : \text{partition}_\Delta(E_1, \ldots, E_\Delta, E) \wedge \tag{7}$$

$$\forall v, w\ \exists E^\star : \Big(\text{path}(v, w, E^\star) \wedge \tag{8}$$

$$\exists X_1, \ldots, X_\Delta : \Big(\text{partition}_\Delta(X_1, \ldots, X_\Delta, V(E^\star) \setminus \{v, w\}) \wedge \tag{9}$$

$$\bigwedge_{1 \le i \le \Delta} \big(\forall x \in X_i\ \exists e_1, e_2 \in E^\star : \big(x = e_1 \cap e_2 \wedge \text{conn}(v, x, E^\star \setminus \{e_1\}) \wedge \tag{10}$$

$$\bigvee_{1 \le i' \le \Delta} (e_1 \in E_{i'} \wedge e_2 \in E_{(i'+i) \bmod \Delta})\big)\big) \wedge \tag{11}$$

$$\bigwedge_{1 \le d \le \text{diam}(G)} \Big(\text{dist}_{d,n+1}(v, w) \Rightarrow \big(\bigvee_{1 \le d' \le \alpha \cdot d} \text{duration}_{d'}(X_1, \ldots, X_\Delta)\big)\Big)\Big)\Big) \tag{12}$$

Note that Lines 7 to 11 have length $\mathcal{O}(\Delta^2)$. Line 12 has length $\mathcal{O}(\alpha \cdot \text{diam}(G)^2 \cdot \Delta \cdot (\alpha \cdot \text{diam}(G))^\Delta)$. Lastly, observe that if $\alpha > \Delta$, we face a trivial yes-instance. Hence, we can estimate the size of $\psi_{G,\Delta,\alpha}$ with $\mathcal{O}((\Delta \cdot \text{diam}(G))^{\Delta+2})$. Lastly, we have that $\psi_{G,\Delta,\alpha}$ can be computed in $\mathcal{O}((\Delta \cdot \text{diam}(G))^{\Delta+2})$ time. Now we have all the ingredients to prove Theorem 26.

Proof of Theorem 26. The correctness of the formula can be shown in an analogous way to Lemma 29. Hence, Theorem 26 follows from Theorem 27 and from the fact that $\psi_{G,\Delta,\alpha}$ is a CMSO-formula with a size in $\mathcal{O}((\Delta \cdot \text{diam}(G))^{\Delta+2})$ that can be computed in $\mathcal{O}((\Delta \cdot \text{diam}(G))^{\Delta+2})$ time. □

7. A Parameterized Local Search Approach

Based on our classical hardness and inapproximability results, it is natural to ask for good polynomial-time heuristic approaches for our problem. From this standpoint, we now consider a 'parameterized local search' version of our problem, that is, we try to improve the stretch of a given labeling by changing the labels of just a few edges. In general, in *parameterized local search* the goal is to improve a given solution by performing a modification that is upper-bounded by k (according to some specified measurement between solutions). Here, k is an additional parameter often referred to as the *search radius* [43]. Marx [36] first considered parameterized local search for the Traveling Salesperson problem in 2008 and since then, these kind of local search problems were considered for many optimization problems (see [36, 24, 27, 43] for some collections of problems).

The following problem we consider in this section generalizes the classical parameterized local search version of SPTGR, as it does not only ask for any improvement but rather for an improvement to some desired stretch α_0. It is formally defined as follows.

Local Search SPTGR (LS SPTGR)

Input:	A graph $G = (V, E)$, $\Delta \in \mathbb{N}$, a labeling $\lambda \colon E \to [1, \Delta]$, $k \in \mathbb{N}$, and a number $\alpha_0 \geq 1$.
Question:	Does there exist a labeling λ' that disagrees with λ on at most k edges, such that the stretch of λ' is at most α_0?

Note that this problem can also be seen as a generalization of SPTGR, as SPTGR is obtained by setting k to the number of edges of the input graph. Generally, our goal is to analyze the parameterized complexity of LS SPTGR with respect to the parameter k. We present an XP-algorithm for k, showing that we can efficiently decide whether a stretch of α_0 can be achieved from our current labeling by changing only a constant number of edge-labels. Note that by Lemma 3, we can also find the optimal stretch in this search space with an additional polynomial factor in the running time. A natural question is then to ask for an FPT algorithm for k. As we show, such an algorithm presumably does not exist, as the problem is W[2]-hard for k, even when asking for any improvement.

Theorem 30. LS SPTGR *admits an XP algorithm when parameterized with k.*

Proof. Our algorithm iterates over all $\mathcal{O}(|E|^k)$ subsets $F \subseteq E$ of size at most k. For each such subset F, we then ask the question, whether we can achieve a stretch of α_0 by changing only the labels of edges of F. That is, we iteratively solve the following intermediate problem.

Fixed Edges Relabel SPTGR	
Input:	A graph $G = (V, E)$, $\Delta \in \mathbb{N}$, a set $F \subseteq E$, a labeling $\lambda' \colon E \setminus F \to [1, \Delta]$, and a number $\alpha_0 \geq 1$.
Question:	Does there exist a labeling $\lambda_F \colon F \to [1, \Delta]$, such that the stretch of $\lambda' \cup \lambda_F$ is at most α_0?

Based on the $|E|^k \cdot n^{\mathcal{O}(1)}$ time (which is XP time) for the iteration over all candidate sets F, to present our XP algorithm, it suffices to show that Fixed Edges Relabel SPTGR admits an XP algorithm for k. Let $e_1, e_2, \ldots, e_k$ be the edges of the given set F, enumerated arbitrarily.

First, assume that $k \neq |E|$, that is, there exists at least one edge of E that is not in F. For every $1 \leq i \leq k$ denote by u_i and v_i the endpoints of the edge e_i; note that some edges of F may have common endpoints. Denote the set of all endpoints of the edges of F by $V_F = \{u_i, v_i : 1 \leq i \leq k\}$. We now define the set $N_F = \bigcup_{v \in V_F} \{e \in E \setminus F : v \in e\}$ of all edges in G that are not in F but share at least one common endpoint with some edge in F. Furthermore, we define the set $L_0 = \{\lambda(e) : e \in N_F\}$ of all distinct time-labels that appear in at least one edge of N_F. Let $\ell_1 < \ldots < \ell_t$ be the labels of L_0 in increasing order. For simplicity of the presentation, we assume without loss of generality that $\ell_1 = 1$; this can be achieved by subtracting $\ell_1 - 1$ from the time-label of every edge. Note that $t = |L_0| \leq |N_F| \leq \sum_{v \in V_F} |N(v)| - |F| \leq k \cdot \deg_{\max} \in \mathcal{O}(n^2)$. Finally, we define the $2t$ subsets $Z_1, \ldots, Z_{2t}$ of time-labels, called *zones*, as follows. For every $j = 1, 2, \ldots, t-1$, we define $Z_{2j-1} = \{\ell_j\}$ and $Z_{2j} = \{\ell_j + 1, \ldots, \ell_{j+1} - 1\}$. For $j = t$, we define $Z_{2t-1} = \{\ell_t\}$ and $Z_{2t} = \{\ell_t + 1, \ldots, \Delta\}$.

For every edge $e_i \in F$, we guess in which zone Z_j the label $\lambda(e_i)$ lies. These are in total $(2t)^k$ different cases, as we have k edges in F and $2t$ potential zones for the label of each edge. Furthermore, for every allocation of the edges of F to the $2t$ different zones of time-labels, we also guess a permutation of the time-labels of the edges of F which are allocated to the same zone. These are in total at most $k!$ different permutations. Each of these permutations corresponds to a different relative order of the time-labels of the edges of F. For each of these permutations, we also guess whether two consecutive time-labels within the same zone are equal or not; these are 2^k different choices. Summarizing, for every allocation of the edges of F to the

$2t$ different zones of time-labels, we guess the relative order of the time-labels of the edges of F in each zone, by distinguishing which time-labels are equal and which are different. We call each of these relative orders a *time-label profile*; there are at most $2^k k!$ different profiles.

Let P be an arbitrary temporal path, and let e_1, e_2 be any two consecutive edges in P, with v being their common endpoint. Note that, once we have fixed an allocation of the edges of F to the $2t$ different zones and a time-label profile, we know the relative order of the time-labels ℓ_1 and ℓ_2 of the edges e_1 and e_2, respectively. More specifically, if $\ell_2 > \ell_1$, then the waiting time at v is $\ell_2 - \ell_1$; if $\ell_2 < \ell_1$, then the waiting time at v is $\Delta + \ell_2 - \ell_1$; if $\ell_2 = \ell_1$, then the waiting time at v is Δ.

Consider two arbitrary vertices z and w in G and an arbitrary edge $e_i = u_i v_i$ of F. Let P be a fastest temporal path P from z to w, and assume that P passes through e_i. Without loss of generality, let P first visit u_i and then v_i. Note that, given a fixed allocation of the edges of F to the $2t$ different zones and a fixed time-label profile, if e_i is neither the first nor the last edge of P, then the duration of P is *independent* of the exact time-label of the edge e_i. The reason is that, in this case, the relative order of the time-label of e_i, compared to the time-labels of the previous and the next edge on P, is fixed in the given time-label profile, regardless of the exact value of the time-label of e_i.

Now suppose that e_i is the *last* (resp. *first*) edge of P, i.e. $w = v_i$ (resp. $w = u_i$). Then, the duration of P is smallest when the time-label of e_i is the smallest (resp. largest) possible, while still respecting the relative order of the time-labels in the given profile.

Note that, if we fix a *specific* time-label for each edge of F, we can trivially compute the stretch α of this specific time-labeling. This can be done by just computing the fastest temporal path from every vertex z to every other vertex w, dividing its duration by the length of the shortest path between z and w in the underlying graph G, and returning the largest of these ratios as the stretch α of this time-labeling.

Our algorithm proceeds as follows, while examining a fixed time-label profile. For each edge $e_i \in F$, we perform binary search for the time-label of e_i in the zone Z_j in which e_i is allocated (independently of any other edge $e_{i'}$ of F), until we find a time-labeling (if it exists) that gives a stretch α that is at most the stretch threshold α_0. During this procedure of performing multiple binary searches on each edge of F independently, we only consider those time-labelings that conform with the current time-label profile.

We iterate over all possible $2^k k!$ different profiles and, for each of them, we perform the above binary searches. If, during this procedure, we detect a time-labeling of the edges of F that gives a stretch $\alpha \leq \alpha_0$, then we return this time-labeling; otherwise, we return that such a labeling does not exist. The running time of this algorithm is

$$(2k \cdot \deg_{\max} \cdot \log \Delta)^k 2^k k! \cdot (n + \log \Delta)^{\mathcal{O}(1)} = (2n^2 \log \Delta)^k 2^k k! \cdot (n + \log \Delta)^{\mathcal{O}(1)},$$

as there are at most $2t \leq 2k \cdot \deg_{\max} \leq 2n^2$ different allocations of each of the k edges of F to a time-label zone, at most $2^k k!$ different profiles, all independent binary searches on the k edges can be performed in $(\log \Delta)^k$ time, which is XP time, since the input size includes $\log \Delta$. Thus this is an XP algorithm for Fixed Edges Relabel SPTGR with respect to the parameter k. By implementing the above algorithm iteratively for each of the $\mathcal{O}(|E|^k)$ subsets $F \subset E$ of k edges, we obtain an XP algorithm for LS SPTGR with respect to k.

Second, assume that $k = |E|$, that is, every edge of the graph is in the set F. Then $N_F = \emptyset$ and $t = 0$. In this case, we create just one time-label zone $Z_1 = \{1, \ldots, \Delta\}$ that contains all possible Δ time-labels, and then we perform exactly the same algorithm as above. As we now have only one zone of time-labels, the running time becomes $(\log \Delta)^k 2^k k! \cdot (n + \log \Delta)^{\mathcal{O}(1)}$. □

We now show that there is presumably no FPT algorithm for LS SPTGR for parameter k; recall that k is part of the input.

Theorem 31. LS SPTGR *is W[2]-hard when parameterized by k.*

Proof. We show that it is W[2]-hard to decide whether a stretch of $\alpha_0 = \alpha - \epsilon$ can be achieved by changing the labels of at most k labels.

We reduce from Hitting Set which is W[2]-hard for k [11].

Hitting Set	
Input:	A universe U, a set of *hyperedges* $\mathcal{F} \subseteq 2^U$, and $k \in \mathbb{N}$.
Question:	Is there a *hitting set* of size at most k for $\mathcal{F}$, that is, a set $S \subseteq U$ of size at most k, such that $S \cap F \neq \emptyset$ for each $F \in \mathcal{F}$?

Let $I := (U, \mathcal{F}, k)$ be an instance of Hitting Set and assume without loss of generality that each hyperedge has size at least k. Let G be the incidence graph of I, that is, the bipartite graph that (i) contains for each element $u \in U$ a vertex u, (ii) contains for each hyperedge $F \in \mathcal{F}$ a vertex v_F,

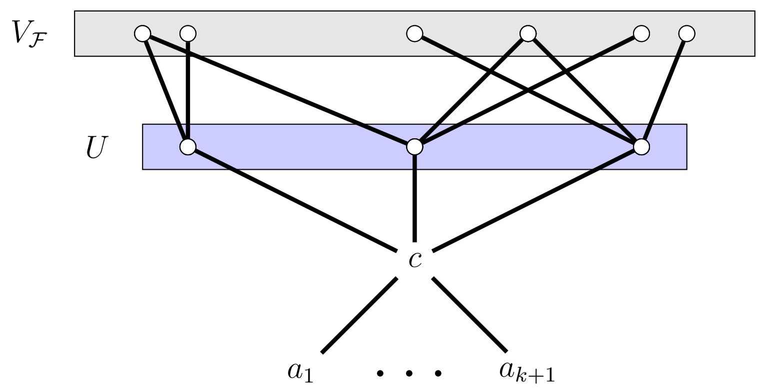

Figure 7: An illustration for the reduction behind Theorem 31 for the instance I of Hitting Set with. All edges receive label 1 under the initial labeling. For each vertex a_i and each vertex $v_F \in V_{\mathcal{F}}$, at least one edge from c to a vertex u with $u \in F$ needs to be relabeled from 1 to 2, to ensure that there is a path of duration less than 5 between a_i and v_F. This then resembles a hitting set.

and (iii) contains an edge $\{u, v_F\}$ with $u \in U$ and $F \in \mathcal{F}$ if and only if $u \in F$. Let $V_{\mathcal{F}} := \{v_F \mid F \in \mathcal{F}\}$. We now extend G as follows: First, we make V_F a clique in G. Next, we add a vertex c to G which we make adjacent to all vertices of U. Finally, we add $k+1$ degree-1 neighbors for c to G. We denote these degree-1 neighbors by a_i with $i \in [1, k+1]$. This completes the definition of G. An illustration of this construction is depicted in Figure 7.

Now, let $\Delta := 2$ and let $\lambda \colon E(G) \to \{1, 2\}$ be the labeling that assigns label 1 to each edge.

Note that the diameter of G is 3 and that the vertex pairs of distance 3 are exactly the pairs of $A \times V_{\mathcal{F}}$ where $A := \{a_i \mid i \in [1, k+1]\}$. Hence, the stretch is $\Delta - \frac{\Delta-1}{3} = \frac{5}{3}$ and can only be improved if for each pair of $A \times V_{\mathcal{F}}$, at least one path has duration less than 5.

We now show that there is a hitting set of size at most k for $\mathcal{F}$ if and only if there is a labeling λ' that changes the labels of at most k edges, such that the stretch of λ' is strictly better than the stretch of λ.

($\Rightarrow$) Let S be a hitting set of size at most k for $\mathcal{F}$. We define a labeling λ' as follows: For each element $u \in S$, we set $\lambda(\{c, u\}) := 2$. We assign label 1 to all other edges. Hence, only the label of at most k edges was changes with respect to λ. It remains to show that the stretch of λ' is better than the stretch of λ. Since the stretch of $\frac{5}{3}$ is only achieved for vertex pairs of $A \times V_{\mathcal{F}}$. We show that for these vertex pairs there is a path of duration 3 under λ', which then implies that the stretch of λ' is better than the stretch of λ.

Let $F \in \mathcal{F}$ and let $a_i \in A$. Since S is a hitting set for $\mathcal{F}$, there is some element $u \in S \cap F$. Hence, by definition of λ', $\lambda'(\{c,u\}) = 2$. Moreover, since $u \in F$, $\{v_F, u\}$ is an edge of G that receives label 1 under λ'. Finally, $\{c, a_i\}$ also receives label 1 under λ. This implies that the path (v_F, u, c, a_i) (and its reverse) has labels $(1,2,1)$ and thus a duration of 3. Consequently, λ' has a stretch strictly better than $\frac{5}{3}$.

($\Leftarrow$) Let $\lambda' \colon E(G) \to \{1,2\}$ be a labeling that achieves a stretch strictly better than $\frac{5}{3}$ while only changing the labels of at most k. Let X denote these edges. This implies that for each distance-3 vertex pair $(a_i, v_F) \in A \times V_F$, there are paths of duration at most 4 between a_i and v_F under λ'. Since $|A| = k+1$, there is at least one edge of $\{\{a_i, c\} \mid i \in [1, k+1]\}$ that still receives label 1 under λ'. Hence, we can assume without loss of generality that $X \cap \{\{a_i, c\} \mid i \in [1, k+1]\} = \emptyset$.

Now, we analyze the structure of the labeling λ'. First, we show that we can assume without loss of generality that X is a subset of the edges of $\{\{c, u\} \mid u \in U\}$.

Let $F \in \mathcal{F}$. Since there is a path under λ' from v_F to a_1 of duration at most 4, we only have to consider two cases.

- Each path P of duration at most 4 between v_F and a_1 has length at least 4 (and thus exactly 4), or
- there is a path P of duration at most 4 and length 3 between v_F and a_1.

In the first case, the path P is of the form $(v_F, v_{F'}, u, c, a_1)$ for a hyperedge $F' \in \mathcal{F}$ distinct from F and an element $u \in F'$. Since P has duration 4 and $\lambda'(\{c, a_1\}) = 1$, P has the labels $(2,1,2,1)$, which implies that $\{v_F, v_{F'}\} \in X$. Let $u' \in F$. Consider the labeling λ'' obtained from λ' by labeling $\{v_F, v_{F'}\}$ to 1 and labeling $\{c, u'\}$ to 2. Hence, the number of changes with respect to λ is at most k and each pair of $A \times V_{\mathcal{F}}$ still admits paths of duration at most 4. This is due to the fact, that (i) for $v_{F'}$, the path $(v_{F'}, u, c, a_1)$ still has labels $(1,2,1)$ and thus a duration of $3 < 4$, (ii) (v_F, u', c, a_1) has now labels $(1,2,1)$ or $(2,2,1)$, which implies a path of duration at most 4, and (iii) only the vertices v_F and $v_{F'}$ can traverse the edge $\{v_F, v_{F'}\}$ on any path of length at most 4 towards a vertex of A.

Hence, after applying this exchange operation exhaustively, we can assume that for each $F \in \mathcal{F}$, there is a path of length 3 and duration at most 4 from v_F to a_1. That is, there is a path $P = (v_F, u, c, a_1)$ for some $u \in F$.

Since this path has duration at most 4, X contains at least one of $\{v_F, u\}$ and $\{u, c\}$ (since $\{c, a_1\} \notin X$). If $\{u, c\} \notin X$, then $\{v_F, u\} \in X$. Consider the labeling λ'' obtained from λ' by labeling $\{v_F, u\}$ with 1 and labeling $\{u, c\}$ with 2. Hence, the number of changes with respect to λ is at most k and each pair of $A \times V_{\mathcal{F}}$ still admits paths of duration at most 4 and length at most 3.

Hence, after exhaustively applying this operation, for each $F \in \mathcal{F}$, there is some $u \in F$ such that $\{c, u\} \in X$. Since X has size at most k, this implies that there is a hitting set of size at most k for $\mathcal{F}$. □

8. Conclusion

In this paper, we investigated a natural temporal graph realization problem, where the durations of the fastest connections in the produced periodic temporal graph shall be at most a multiplicative factor times the corresponding distances in the static graph. Among other results, we showed that the problem is hard to solve exactly and also hard to approximate within a constant factor on general instances. We also designed a polynomial time algorithm for general graphs and that achieves a factor-2 approximation on trees, whereas there are NP-hard instances for which the guaranteed stretch is tight. Our work leaves several natural future work directions: Are there instances where the optimal stretch is strictly larger than $\frac{\Delta+1}{2}$? What is the complexity of SPTGR for $\Delta = 2$? Can we identify larger graph classes than trees, where we can achieve a constant-factor approximation? What is the parameterized complexity of SPTGR with respect to structural parameters of the input graph (independent of Δ, e.g. treewidth by itself)? What types of results can we achieve when we allow an additive stretch, or if we want to minimize the *average* stretch?

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Appendix A. Deferred Proofs

For the sake of completeness, we provide the proofs of Claim 20.1 and Lemma 21, thus verifying the correctness of Theorem 18.

Appendix A.1. Proof of Claim 20.1

Proof of Claim. To prove the statement, we show the existence of two special kinds of fast paths:

- For each $v \in V \cup Y$, there is a vertex $x \in X$ that is part of the same sunglasses gadget of v and a path P_v of duration at most $\Delta - 1$ from x to v that starts with an edge of label ℓ_3.

- For each $v \in V \cup Y \cup X$ and each distinct vertex $x \in X$, there is a path $P_{v,x}$ from v to x with (i) duration at most ℓ_3 that ends with an edge of label ℓ_2 or (ii) duration at most ℓ_2 that ends with an edge of label ℓ_1.

First, we show the existence of the first type of fast paths. Let $\overline{e} = \{u, w\}$ be the non-edge of G for which v is a vertex of the sunglasses gadget $S_{\overline{e}}$. Since v is not a vertex of X, we can assume without loss of generality that v is part of the chronological cycle $C^u_{\overline{e}}$. We set x to be the unique vertex of X in $S_{\overline{e}}$ that is part of the chronological (w, u)-path. By definition of the sunglasses labeling, x has a neighbor s in $C^u_{\overline{e}}$, such that edge $\{s, x\}$ receives label ℓ_3 and the edge of s towards its successor in $C^u_{\overline{e}}$ has label ℓ_3+1. Consider the path P_v that starts at x, moves to s, and then follows the chronological cycle $C^u_{\overline{e}}$ until reaching vertex v. Since Δ is even, $C^u_{\overline{e}}$ contains $\Delta - 1$ vertices. Hence, P_v contains at most Δ vertices and thus $\Delta - 1$ edges. By choice of x, the labels of P_v increase by exactly one (modulo Δ) for consecutive edges. This implies that P_v has duration of at most $\Delta - 1$. Moreover, if $v \in V$, then the last edge of P_v has label Δ.

Next, we show the existence of the second type of fast paths. Let $v \in V \cup Y \cup X$ and each distinct vertex $x \in X$. We distinguish three cases:

Case: $v \in X$**.** Then $\{v, x\}$ is an edge of label ℓ_2, which proves the existence of the claimed path.

Case: $v \in Y$**.** By definition of the sunglasses labeling, there is a zigzag path Z from v to some vertex $x' \in X$ which is part of the same chronological cycle, such that Z has duration at most ℓ_1 and reaches vertex x' with an edge of label ℓ_1. If $x = x'$, this shows the existence of the claimed path. Otherwise, further traverse the edge $\{x', x\}$ with label ℓ_2. This then implies a path of duration at most ℓ_2 that reaches x with an edge of label ℓ_2, which proves the existence of the claimed path. Note that in both cases, the duration of the presented path to x is by one lower than the stated duration. We will make use of that additional duration 1 in the following final case.

Case: $v \in V$**.** If $v \in V$, then there is a chronological cycle C attached to v, and v has an incident edge in C of label Δ. Let s denote the endpoint of this edge. The above discussed zigzag path Z from s to some vertex of X starts with label 1. Hence, by going from v to s at time Δ and then following the zigzag path from s to x, we obtain a desired path from v to some vertex of X, since it only increases the proven duration of the path from s to x by one. In particular, note that this path starts with label Δ. This shows the existence of the second type of fast paths.

Based on the existence of these types of fast paths, we can now prove the claim. Let $v \in V \cup Y \cup X$ and let $w \in V \cup Y \cup X$. Due to symmetry, we only present a fast path from v to w. Since both v and w are vertices of at least one sunglasses gadget, the above implies that there is a path P_w of duration at most $\Delta - 1$ from some vertex $x \in X$ to w. Moreover, the label of the first edge of P_w is ℓ_3. Additionally, by the above, there is a path $P_{v,x}$ from v to x with (i) duration at most ℓ_3 that ends with an edge of label ℓ_2 or (ii) duration at most ℓ_2 that ends with an edge of label ℓ_1. In both cases, the duration of the path P from v to w obtained from first following $P_{v,x}$ and then following P_w is at most $\ell_3 + \Delta - 1 = \ell_2 + \Delta = 3 \cdot \frac{\Delta}{2}$. Moreover, if $v \in V$, then the first label of P is Δ, since the first label of $P_{v,x}$ is Δ. Finally, if $w \in V$, then the last label of P is Δ, since the last label of P_w is Δ. ■

Appendix A.2. Proof of Lemma 21

Proof. Recall that $V^* = \{v_1^*\}$ since $\Delta = 4$. For better readability, denote $v^* := v_1^*$. Further, recall that v^* is a degree-1 neighbor of c. Moreover, the neighborhood of c is $V \cup \{c_1, c_2, c_3\} \cup \{v^*\}$, the neighborhood of each vertex of c_i is $\{c, c_1, c_2, c_3\} \cup X$, and each vertex of Y is adjacent to at least one vertex of V and at least one vertex of X. In other words, no vertex of V' has distance more than 2 (3) with $\{c, c_1, c_2, c_3\}$ (v^*), the vertices of distance exactly 2 (3) with c (v^*) are the vertices of $X \cup Y$, the vertices of distance exactly 2 with v^* are the vertices of $V \cup \{c_1, c_2, c_3\}$, and the vertices of distance 2 with any vertex of $\{c_1, c_2, c_3\}$ are the vertices of $V \cup Y$.

First, we consider the vertex $q = v^*$. By definition of λ, the edge $\{v^*, c\}$ receives label 4. This is the incident with c of label 4. Hence, c is a nice neighbor of v^* and each other neighbor of c, that is, for each vertex $z \in V \cup \{c_1, c_2, c_3\}$, c is a nice neighbor of v^* and vertex z. This implies that the statement hold for all vertices $z \in V'$ with distance at most two with v^*. Hence, consider a vertex z of distance exactly 3 with v^*. By the initial argument on the distances in G', z is a vertex of Y. Moreover, by definition of the sunglasses gadgets and the sunglasses labeling, there is a vertex $x \in X$ in the same sunglasses gadget as z, such that the edge between z and x is part of a zigzag path (of length one) and thus receives label $\ell_1 = 1$. This holds, since $\Delta = 4$. Consider the path $P_{v^*,z} := (v^*, c, c_2, x, z)$. By definition of λ, the labels of the edges of this path are $4, 2, 4, 1$. Hence, $P_{v^*,z}$ has a duration of $6 = 3 \cdot \frac{\Delta}{2} = \frac{\Delta}{2} \cdot D_{v^*,z}$. Moreover, note that the subpath (c, c_2, x, z) has a duration of $4 = \Delta = \frac{\Delta}{2} \cdot D_{c,z}$. Now, consider the path $P_{z,v^*} := (z, x, c_3, c, v^*)$. By definition of λ, the labels of the edges of this path are $1, 2, 3, 4$. Hence, $P_{v^*,z}$ has a duration of $4 < 3 \cdot \frac{\Delta}{2} = \frac{\Delta}{2} \cdot D_{v^*,z}$. Moreover, note that the subpath (z, x, c_3, c) has a duration of $3 < \Delta = \frac{\Delta}{2} \cdot D_{c,z}$. Hence, the statement holds for $q = v^*$.

Next, we consider the case $q = c$. Recall that the only vertices of distance more than 1 with c are the vertices of $X \cup Y$. As argued by the above subpaths, for each vertex y of Y, there are paths of duration at most $\frac{\Delta}{2} \cdot D_{c,z}$ each between c and v. Hence, we only have to consider the durations of paths between c and vertices of X. For each such vertex $x \in X$, $\lambda(\{c, c_1\}) = 1 \neq 4 = \lambda(\{c_1, x\})$. That is, c_1 is a nice neighbor of both c and x. This implies that there are paths between c and x of duration at most $\Delta = \frac{\Delta}{2} \cdot D_{c,x}$ each. Hence, the statement holds for $q = v^*$.

It remains to consider the case of $q \in \{c_1, c_2, c_3\}$. Recall that the only vertices of distance more than 1 with q are the vertices of $V \cup Y$. Let $z \in Y$. As discussed before, there is a vertex $x \in X$ in the same sunglasses gadget as z, such that the edge between z and x is part of a zigzag path (of length one) and thus receives label $\ell_1 = 1$. This holds, since $\Delta = 4$. Hence, $\lambda(\{z, x\}) = 1 \neq \lambda(\{x, q\}) \in \{2, 4\}$. That is, x is a nice neighbor of both z and q. This implies that there are paths between c and x of duration at most $\Delta = \frac{\Delta}{2} \cdot D_{c,x}$ each. Finally, let $z \in V$. Moreover, let $i \in \{1, 2, 3\}$, such that $q = c_i$. Note that c is a common neighbor of v_i and z and that $\lambda(\{c, c_i\}) = i$. If $\lambda(\{z, c\}) \neq i$, then c is a nice neighbor of z and c_i. In this case, the statement holds. Thus, assume that $\lambda(\{z, c\}) = i$. We distinguish three cases.

Case: $i = 1$**.** Consider the paths $P_{c_i,z} := (c_1, c_3, c, z)$ and $P_{z,c_i} := (z, c, c_2, c_1)$. The labels of the edges of these paths are $2, 3, 1$, and $1, 2, 3$, respectively.

Case: $i = 2$**.** Consider the paths $P_{c_i,z} := (c_2, c_1, c, z)$ and $P_{z,c_i} := (z, c, c_3, c_2)$. The labels of the edges of these paths are $3, 1, 2$, and $2, 3, 1$, respectively.

Case: $i = 3$**.** Consider the paths $P_{c_i,z} := (c_3, c_2, c, z)$ and $P_{z,c_i} := (z, c, c_1, c_3)$. The labels of the edges of these paths are $1, 2, 3$, and $3, 1, 2$, respectively.

In all three cases, the described paths between c_i and z have duration at most $4 = \Delta = \frac{\Delta}{2} \cdot D_{c_i,z}$ each. Thus, the statement also holds for $q \in \{c_1, c_2, c_3\}$. This completes the proof. □