Linear Programming Complementation

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Abstract

In this paper we introduce a new operation for Linear Programming (LP), called *LP complementation*, which resembles many properties of LP duality. Given a maximisation (resp. minimisation) LP P, we define its *complement* Q as a specific minimisation (resp. maximisation) LP which has the *same* objective function as P. Our central result is the LP complementation theorem, that relates the optimal value $\textsc{Opt}(P)$ of P and the optimal value $\textsc{Opt}(Q)$ of its complement by $\frac{1}{\textsc{Opt}(P)} + \frac{1}{\textsc{Opt}(Q)} = 1$. The LP complementation operation can be applied if and only if P has an optimum value greater than 1.

To illustrate this, we first apply LP complementation to *hypergraphs*. For any hypergraph H, we review the four classical LPs, namely *covering* $K(H)$, *packing* $P(H)$, *matching* $M(H)$, and *transversal* $T(H)$. For every hypergraph $H = (V, E)$, we call $\overline{H} = (V, \{V \setminus e : e \in E\})$ the *complement* of H. For each of the above four LPs, we relate the optimal values of the LP for the dual hypergraph H^* to that of the complement hypergraph $\overline{H}$ (e.g. $\frac{1}{\textsc{Opt}(K(H^*))} + \frac{1}{\textsc{Opt}(K(\overline{H}))} = 1$).

We then apply LP complementation to *fractional graph theory*. We prove that the LP for the *fractional in-dominating number* of a digraph D is the complement of the LP for the *fractional total out-dominating number* of the digraph complement $\overline{D}$ of D. Furthermore we apply the hypergraph complementation theorem to matroids. We establish that the fractional matching number of a matroid coincide with its edge toughness.

As our last application of LP complementation, we introduce the natural problem Vertex Cover with Budget (VCB): for a graph $G = (V, E)$ and a positive integer b, what is the maximum number t_b of vertex covers $S_1, \ldots, S_{t_b}$ of G, such that every vertex $v \in V$ appears in at most b vertex covers? The integer b can be viewed as a "budget" that we can spend on each vertex and, given this budget, we aim to cover all edges for as long as possible. We relate VCB with the LP Q_G for the fractional chromatic number χ_{f} of a graph G. More specifically, we prove that, as $b \to \infty$, the optimum for VCB satisfies $t_b \sim t_{\mathrm{f}} \cdot b$, where t_{f} is the optimal solution to the complement LP of Q_G. Finally, our results imply that, for any finite budget b, it is NP-hard to decide whether $t_b \geq b + c$ for any $1 \leq c \leq b - 1$.

1 Introduction

1.1 Background

Many optimisation problems can be expressed as, or reduced to, Linear Programs (LPs) or Integer Programs (IPs) [11]. As such, the use of Linear Programming is ubiquitous [14], with applications in combinatorial optimisation, combinatorics, industrial engineering, coding theory, etc. One of the key aspects of Linear Programming is LP duality, and in particular the strong LP duality theorem which states that the optimal value of an LP is equal to that of its dual [13].

Many classical problems from graph theory, e.g. maxmimum matching, minimum vertex cover, chromatic number, independence number, clique number, minimum dominating set, domatic number, etc. can be expressed as Integer Programs (IPs). Fractional graph theory then investigates these problems with three main approaches (see the book by Scheinerman and Ullman [12] for a survey).

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First, it studies the Linear Programming (LP) relaxations of these IPs, some of which have found applications of their own, e.g fractional chromatic number or fractional domatic number for scheduling [1, 5]. Second, it applies LP techniques to either the original IP problems or their LP relaxations. Amongst those, LP duality is one of the most powerful and ubiquitous [13]. Third, it generalises the results to hypergraphs in order to get a clearer framework. In particular, hypergraph duality, where the roles of vertices and edges are swapped, is common practice.

Motivating example. We illustrate the main contributions of this paper, namely LP and hypergraph complementations, via a simple example first. A *vertex cover* of a graph is a set of vertices that includes at least one endpoint of every edge of the graph. A *stable set* of a graph is a set of pairwise non-adjacent vertices. Consider the following problem VERTEX COVER WITH BUDGET (VCB). Given a graph G and a vertex budget $b \geq 1$, find the largest collection $S_1, \ldots, S_{t_b}$ of vertex covers such that every vertex belongs to at most b of the S_i's. As b tends to infinity, the optimum satisfies $t_b \sim t_\mathrm{f} \cdot b$, where t_f is defined as follows. Let A be the incidence matrix of vertex covers of G, where $A_{ve} = 1$ if and only if v belongs to the vertex cover e. Then $t_\mathrm{f} = \max\{1^\top x : Ax \leq 1, x \geq 0\}$. For instance, the pentagon C_5 has eleven vertex covers: all complements of non-edges and all sets of four or five vertices. We obtain

$$A(C_5) = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \qquad t_\mathrm{f}(C_5) = \max\{1^\top x : Ax \leq 1, x \geq 0\} = \frac{5}{3}.$$

In general, the t_f quantity has received very little attention. LP complementation then links it to the much more well studied fractional chromatic number of graphs [12]. A c-multicolouring of G is the smallest size of a collection of stable sets $\overline{S}_1, \ldots, \overline{S}_{\chi_c}$, such that each vertex belongs to at least c of the $\overline{S}_i$'s. As c tends to infinity, the optimum satisfies $\chi_c \sim \chi_\mathrm{f} \cdot c$, where χ_f is the fractional chromatic number of G. We have $\chi_\mathrm{f} = \min\{1^\top x : (1 - A)x \geq 1, x \geq 0\}$, where $1 - A$ is the incidence matrix of stable sets of G. For instance, for C_5,

$$1 - A(C_5) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \chi_\mathrm{f}(C_5) = \min\{1^\top x : (1 - A)x \geq 1, x \geq 0\} = \frac{5}{2}.$$

The relation between vertex covers and stable sets is an example of hypergraph complementation, and accordingly, the t_f and χ_f terms are examples of LP complementation. We shall prove that these two values form a complement pair, i.e. $\frac{1}{t_\mathrm{f}} + \frac{1}{\chi_\mathrm{f}} = 1$, which is easily verified for C_5. Therefore, computing the fractional chromatic number immediately yields the asymptotic behaviour of the vertex cover with budget problem.

1.2 Our contributions

In this subsection, we give an overview of the contributions of this paper. Here, we provide a selected number of definitions and simplified statements of results that shall be proved in the rest of the paper.

1.2.1 Linear Programming Complementation

In this paper we introduce the notion of the *complement* of an LP R, which we denote by $\overline{R}$, as follows. Let $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$, then for the following maximisation LP P, we have

$$P: \ \max\{c^\top x : Ax \leq b\},$$
$$\overline{P}: \ \min\{c^\top x : (bc^\top - A)x \geq b\}.$$

Similarly, let $v \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $M \in \mathbb{R}^{m\times n}$, then for the following minimisation LP Q, we have

$$Q: \ \min\{v^\top x : Mx \geq u\},$$
$$\overline{Q}: \ \max\{v^\top x : (uv^\top - M)x \leq u\}.$$

To simplify notation, in the remainder of the paper we use the notation P (resp. Q) to denote a maximisation (resp. minimisation) LP, while we use R to denote an arbitrary LP which can be either a maximisation or a minimisation LP. Furthermore, for any linear program R, adding the constraint that the variables have to be integral yields an integer program, which we denote $R^{\mathbb{Z}}$.

LP complementation theorem. Our central result is a surprising relation between the optimal values of an LP and its complement, given that one of these values is finite and larger than 1.

Theorem 1.1 (LP complementation theorem)**.** *For any LP R, $1 < \textsc{Opt}(R) < \infty$ if and only if $1 < \textsc{Opt}(\overline{R}) < \infty$, in which case*

$$\frac{1}{\textsc{Opt}(R)} + \frac{1}{\textsc{Opt}(\overline{R})} = 1.$$

Alternatively, the theorem states that the harmonic mean of the optimal values of the LP and its complement is 2. Consequently, the two values are separated by 2, and one value is equal to 2 if and only if the other is equal to 2.

Natural interpretation of LP complementation. The links between two-player zero-sum (matrix) games and LP are well established; see [4,15] for instance. We shall review these and then show that LP complementation can be interpreted using two complementary games.

Given any $m \times n$ matrix A, the matrix game Γ_A with payoff matrix A is played by two persons, Rose and Colin, as follows. Rose selects a row of A, Colin a column. If the row i and the column j are chosen, then Rose's payoff is a_{ij}. In particular, if $a_{ij} > 0$, then Rose earns money; otherwise, Rose loses money.

A strategy for Colin is then a probability distribution on the columns: $c = (c_1, \ldots, c_n)^\top$ such that $c \geq 0$ and $1^\top c = 1$. Rose's expected payoff for a given strategy c for Colin is then $v_c = \max_i\{A_i c\}$, where A_i is the i-th row of A_i; thus $Ac \leq v_c \cdot 1$. Colin aims at minimising Rose's expected payoff. The value of the game, denoted as V, is the minimum expected of Rose's payoff over all strategies for Colin.

Without loss of generality, suppose that $0 \leq A \leq 1$. Then the value V of the game is also between 0 and 1; let us omit the two extreme cases and suppose that $0 < V < 1$. For any strategy c for Colin with payoff v_c, let $x = \frac{1}{v_c}c$, then we have $x \geq 0$, $Ax \leq 1$, and $1^\top x = \frac{1}{v_c}$. Minimising Rose's expected payoff v_c then corresponds to maximising $1^\top x$. We can then express $V = 1/\textsc{Opt}(P)$, where

$$P: \ \max\{1^\top x : Ax \leq 1, x \geq 0\}.$$

LP duality then corresponds to taking Rose's point of view: $V = 1/\textsc{Opt}(P^*)$, with

$$P^*: \ \min\{1^\top y : A^\top y \geq 1, y \geq 0\}.$$

LP complementation, on the other hand, corresponds to taking the complementary payoff. Consider a second game, where the players change their roles (Rose chooses columns of the payoff matrix and Colin chooses rows), and the payoff is equal to 1 minus the original payoff. Thus, the new payoff matrix is $(1 - A^\top)$ and the value of the new game is $\overline{V} = 1 - V$. But then, we have $\overline{V} = 1/\textsc{Opt}(Q)$, where

$$Q = \overline{P}: \;\; \min\{1^\top y : (1 - A)y \geq 1, y \geq 0\}.$$

We then have $\textsc{Opt}(P) > 1$ and $\textsc{Opt}(\overline{P}) > 1$ and

$$\frac{1}{\textsc{Opt}(P)} + \frac{1}{\textsc{Opt}(\overline{P})} = 1.$$

Consequence for integer programming & Bounds. Let P be a maximisation LP. LP duality can be naturally used to study P, as any feasible solution to the dual P^* gives an upper bound on the optimal value of P. However, a feasible solution to the dual does not provide much information about feasible solutions of the primal. LP complementation works differently, as a feasible solution to the complement immediately yields a feasible solution to the primal by simple scaling. However, it only gives a lower bound on the optimal value. The primal and its complement then "work together" towards their optimal solutions and values.

The relationship between feasible solutions to the primal and the complement has some important consequences for IPs. Firstly, from P and $\overline{P}$, we obtain four programs P_s, ${P_s}^{\mathbb{Z}}$, $(\overline{P})_t$, and ${(\overline{P})_t}^{\mathbb{Z}}$ (where s and t come from an optimal solution of P and its value), which have a common optimal solution–see Corollary 2.4.

Secondly, we introduce the bounds $\alpha(P^{\mathbb{Z}})$ and $\beta(\overline{P}^{\mathbb{Z}})$ on the optimal values of P and $\overline{P}$, respectively. These bounds are based on feasible solutions of $P^{\mathbb{Z}}$ and $\overline{P}^{\mathbb{Z}}$, respectively. We then prove that these bounds are "mutually tight" for the primal-complement pair (they are actually tight for the vertex cover with budget problem on C_5).

Theorem 1.2. *Let $P : \max\{c^\top x : Ax \leq b\}$, where $b > 0$ and $A \neq 0$, such that $1 < \textsc{Opt}(P^{\mathbb{Z}}) \leq \textsc{Opt}(\overline{P}^{\mathbb{Z}}) < \infty$. Then $\textsc{Opt}(P^{\mathbb{Z}}) \leq \alpha(P^{\mathbb{Z}}) \leq \textsc{Opt}(P)$, $\textsc{Opt}(\overline{P}) \leq \beta(\overline{P}^{\mathbb{Z}}) \leq \textsc{Opt}(\overline{P}^{\mathbb{Z}})$ and*

$$\textsc{Opt}(P) = \alpha(P^{\mathbb{Z}}) \iff \textsc{Opt}(\overline{P}) = \beta(\overline{P}^{\mathbb{Z}}).$$

Hypergraph complementation. For a hypergraph $H = (V, E)$ with n vertices and m edges, its *incidence matrix* is denoted by $M_H \in \mathbb{R}^{n \times m}$. The *dual* of the hypergraph H is $H^* = (E, V^*)$, where $V^* = \{E_v : v \in V\}$ and $E_v = \{e \in E : v \in e\}$. We then have $M_{H^*} = (M_H)^\top$ and $(H^*)^* \cong H$.

Now we define the *complement* of H as $\overline{H} = (V, \{V \setminus e : e \in E\})$; note that $M_{\overline{H}} = 1 - M_H$. Hypergraph complementation is an involution that commutes with duality, i.e. $\overline{\overline{H}} = H$ and $\overline{(H^*)} = (\overline{H})^*$.

A *covering* of a hypergraph H is a set of edges whose union is equal to its set of vertices V. The *covering number* $k(H)$ of H is the minimum size of a covering of H; this can be formulated as the optimum of an integer program. The *fractional covering number* $k_{\mathrm{f}}(H)$ of H is the optimal value of the LP $K(H)$ that is obtained by removing the integrality constraints.

It can be easily shown that $\overline{K(H^*)} = K(\overline{H})^*$. By applying the LP complementation theorem to $K(H)$, we obtain the hypergraph complementation theorem, as follows.

Theorem 1.3 (Hypergraph complementation theorem)**.** *For any hypergraph H,*

$$\frac{1}{k_{\mathrm{f}}(H^*)} + \frac{1}{k_{\mathrm{f}}(\overline{H})} = 1.$$

Applying LP duality and hypergraph duality to $K(H)$ yields four standard LPs $K(H), P(H), T(H), M(H)$ for hypergraphs, given in Table 1 and related in Figure 1 [12]. By applying LP complementation to these four LPs, we obtain the four new LPs $\overline{K(H)}, \overline{P(H)}, \overline{T(H)}, \overline{M(H)}$. The new notions of LP complementation and hypergraph complementation allow us to establish a formal relation of these four LPs with the four original LPs; see Figure 2 for an illustration.

Covering number, $k_{\mathrm{f}}(H)$ min # edges to cover all vertices $K(H): \ \min\{1^\top x : M_H x \geq 1, x \geq 0\}$	**Packing number**, $p_{\mathrm{f}}(H)$ max # vertices, no two in the same edge $P(H): \ \max\{1^\top x : M_H^\top x \leq 1, x \geq 0\}$
Transversal number, $\tau_{\mathrm{f}}(H)$ min # vertices to touch all edges $T(H): \ \min\{1^\top x : M_H^\top x \geq 1, x \geq 0\}$	**Matching number**, $\mu_{\mathrm{f}}(H)$ max # pairwise disjoint edges $M(H): \ \max\{1^\top x : M_H x \leq 1, x \geq 0\}$

Table 1: Four standard LPs for hypergraphs: covering, packing, transversal, and matching numbers of a hypergraph.

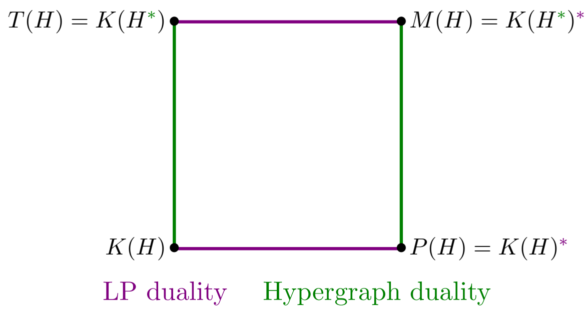

Figure 1: The four initial Linear Programs related to a hypergraph.

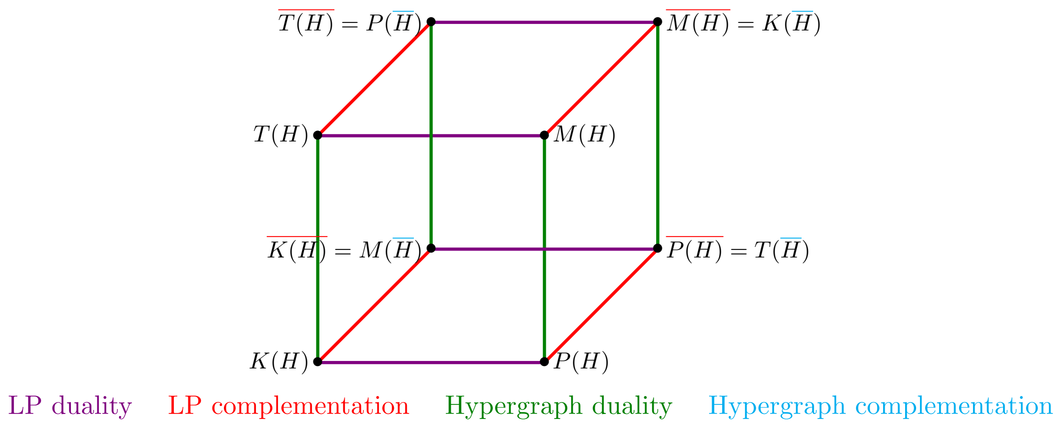

Figure 2: The eight Linear Programs related to a hypergraph.

The hypergraph complementation theorem then holds for all four parameters in Table 1.

Corollary 1.4. *For any hypergraph H, we have*

$$\frac{1}{k_{\mathrm{f}}(H^*)} + \frac{1}{k_{\mathrm{f}}(\overline{H})} = \frac{1}{p_{\mathrm{f}}(H^*)} + \frac{1}{p_{\mathrm{f}}(\overline{H})} = \frac{1}{\mu_{\mathrm{f}}(H^*)} + \frac{1}{\mu_{\mathrm{f}}(\overline{H})} = \frac{1}{\tau_{\mathrm{f}}(H^*)} + \frac{1}{\tau_{\mathrm{f}}(\overline{H})} = 1.$$

1.2.2 The impact of LP complementation to related problems

Here we give a brief overview of the implications that LP Complementation has in the following two case studies. Full details are given in Sections 4 and 5, respectively.

Case study 1: Fractional graph theory. We give two applications of the hypergraph complementation theorem to graph theory.

Firstly, fractional domination in digraphs provides a setting where LP complementation and hypergraph complementation naturally arise. An *in-dominating set* of a digraph D is a set S of vertices such that for any vertex $v \in V(D)$, either $v \in S$ or there exists $s \in S$ such that $(s, v) \in E(D)$; similarly a total in-dominating set is a set T of vertices such that for any vertex v, there exists $t \in T$ such that $(t, v) \in E(D)$. *Out-dominating* and *total out-dominating* sets are defined similarly. The in-dominating number is the smallest cardinality of an in-dominating number; as expected, the total out-dominating number is the smallest cardinality of a total out-dominating set. We relate the *fractional in-dominating number* of a digraph D and the *fractional total out-dominating number* of its digraph complement $\overline{D}$ as follows.

Theorem 1.5 (Domination complementation theorem)**.** *For any digraph D, we have that $\frac{1}{{\gamma^{\rm in}}_{\rm f}(D)} + \frac{1}{{\Gamma^{\rm out}}_{\rm f}(\overline{D})} = 1$.*

This theorem is very general, as it holds for all digraphs, and provides more specific relations about domination numbers for graphs, tournaments, and regular digraphs. The last one is itself a generalisation of the result in [12, Theorem 7.4.1], which only applies to regular *graphs*.

Secondly, we apply the hypergraph complementation theorem to matroids. We establish that the fractional matching number of a matroid coincides with its edge toughness. This result can then be applied to graphic matroids, yielding a formula for the edge toughness of a graph. Moreover, we derive an alternative proof of the relationship between the edge toughness of a matroid and the fractional covering number of its dual matroid.

Case study 2: Vertex cover with budget. We further investigate the Vertex Cover with Budget problem. First, using our LP complementation results we relate the "time per budget" ratio $t_{\rm f}$ to the fractional chromatic number $\chi_{\rm f}$ of the graph by $\frac{1}{t_{\rm f}} + \frac{1}{\chi_{\rm f}} = 1$. Second, we show that, surprisingly, for any finite budget we can also relate the optimal time with multicolourings of the graph. Finally, we prove that, computing an optimum solution, where the budget is finite, is NP-complete.

The rest of the paper is organised as follows. Section 2 first gives the LP complementation theorem. It then investigates its consequences to IP and derives the α and β bounds. In Section 3, we introduce the complement of a hypergraph and apply the LP complementation theorem to obtain the hypergraph complementation theorem. In Section 4, we apply the hypergraph complementation theorem to obtain general results on the fractional dominating number of digraphs and to obtain a new proof of a result on the edge toughness of matroids. Finally, Section 5 applies our results from Sections 2 and 3 to the VCB problem.

2 Linear Programming complementation

2.1 The LP complementation theorem

For any linear program (LP) R which is feasible and bounded, we denote its optimal value as $\textsc{Opt}(R)$. If P is a maximisation problem, then we denote $\textsc{Opt}(P) = -\infty$ if P is infeasible and $\textsc{Opt}(P) = \infty$ if P is unbounded. Similarly, if Q is a minimisation problem, then we denote $\textsc{Opt}(Q) = \infty$ if Q is infeasible and $\textsc{Opt}(Q) = -\infty$ if Q is unbounded. We denote the all-zero vector or matrix as 0, regardless its dimension; similarly, the all-ones vector or matrix is denoted as 1.

We define the *complement* of an LP R, which we denote $\overline{R}$, as follows. Let $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, then for the following maximisation LP P, we have

$$\begin{aligned} P &: \max\{c^\top x : Ax \le b\}, \\ \overline{P} &: \min\{c^\top x : (bc^\top - A)x \ge b\}. \end{aligned}$$

Similarly, let $v \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $M \in \mathbb{R}^{m \times n}$, then for the following minimisation LP Q, we have

$$\begin{aligned} Q &: \min\{v^\top x : Mx \ge u\}, \\ \overline{Q} &: \max\{v^\top x : (uv^\top - M)x \le u\}. \end{aligned}$$

The definition above is extended to general LPs in Table 2.

Primal R	Complement $\overline{R}$
$\max c^\top x$	$\min c^\top x$
$A_l x \leq b_l$	$(b_l c^\top - A_l)x \geq b_l$
$A_e x = b_e$	$(b_e c^\top - A_e)x = b_e$
$A_g x \geq b_g$	$(b_g c^\top - A_g)x \leq b_g$
$x_l \leq 0$	$x_l \leq 0$
$x_g \geq 0$	$x_g \geq 0$
x_f free	x_f free

Table 2: General definition of LP complement.

Complementation is an involution, i.e. $R = \overline{\overline{R}}$. Moreover, complementation commutes with duality: indeed, if R^* denotes the dual of R, then we have $\overline{(R^*)} = (\overline{R})^*$.

Say two real numbers $x, y > 1$ are a *complement pair* if $\frac{1}{x} + \frac{1}{y} = 1$. The main result is that, provided $1 < \textsc{Opt}(R) < \infty$ or $1 < \textsc{Opt}(\overline{R}) < \infty$, then the optimal values of R and $\overline{R}$ form a complement pair.

Theorem 2.1 (LP complementation theorem)**.** *For any LP R, $1 < \textsc{Opt}(R) < \infty$ if and only if $1 < \textsc{Opt}(\overline{R}) < \infty$, in which case*

$$\frac{1}{\textsc{Opt}(R)} + \frac{1}{\textsc{Opt}(\overline{R})} = 1.$$

Proof. Without loss of generality, let $P : \max\{c^\top x : Ax \leq b\}$. Suppose $1 < \textsc{Opt}(P) < \infty$, say $\textsc{Opt}(P) = 1 + a$ for some $a > 0$. Let x be an optimal solution of P, and let $\overline{x} = \frac{1}{a}x$. We then have

$$(bc^\top - A)\overline{x} = \frac{1+a}{a}b - \frac{1}{a}Ax \geq b,$$

and hence $\overline{x}$ is a feasible solution of $\overline{P}$, with value $1 + \frac{1}{a}$.

We have just shown that $\overline{P}$ has a feasible solution of value greater than one. We now prove that $\textsc{Opt}(\overline{P}) > 1$. For the sake of contradiction, suppose that $\overline{P}$ has a feasible solution with value at most 1, then for any $\epsilon > 0$, $\overline{P}$ has a feasible solution $\overline{y}$ with value $1 + \epsilon$. Let $y = \frac{1}{\epsilon}\overline{y}$, then by the same reasoning as above, y is a feasible solution of P with value $1 + \frac{1}{\epsilon}$; we conclude that P is unbounded, which is the desired contradiction.

Having established that $1 < \textsc{Opt}(\overline{P}) < \infty$, we find that the first paragraph showed that

$$\frac{1}{\textsc{Opt}(P)} + \frac{1}{\textsc{Opt}(\overline{P})} \geq \frac{1}{a+1} + \frac{a}{a+1} = 1.$$

We now prove the reverse inequality. Let $\textsc{Opt}(\overline{P}) = 1 + \overline{a}$ with $\overline{a} > 0$ and $\overline{x}$ be an optimal solution of $\overline{P}$. Then $x = \frac{1}{\overline{a}}\overline{x}$ is a feasible solution of P with value $1 + \frac{1}{\overline{a}}$, and we obtain

$$\frac{1}{\textsc{Opt}(P)} + \frac{1}{\textsc{Opt}(\overline{P})} \leq \frac{\overline{a}}{\overline{a}+1} + \frac{1}{\overline{a}+1} = 1.$$

The case where we suppose $1 < \textsc{Opt}(\overline{P}) < \infty$ instead is similar and hence omitted. □

Observation 2.2. *If (a, b) form a complement pair, with $a \leq b$, then $a \leq 2 \leq b$. Moreover, the following are equivalent: $a = 2$; $b = 2$; $a = b$.*

The LP complementation theorem then has this immediate consequence.

Corollary 2.3. *Suppose $1 < \textsc{Opt}(P) \leq \textsc{Opt}(\overline{P}) < \infty$. Then*

$$\textsc{Opt}(P) \leq 2 \leq \textsc{Opt}(\overline{P}).$$

Moreover, the following are equivalent: $\textsc{Opt}(P) = 2$; $\textsc{Opt}(\overline{P}) = 2$; $\textsc{Opt}(P) = \textsc{Opt}(\overline{P})$.

2.2 Feasibility and boundedness

The strong duality theorem not only states that the optimal values of a primal LP and that of its dual are equal whenever they are finite, but it also considers the case of infeasibility and unboundedness: if an LP is unbounded, then its dual is infeasible; if the dual is unbounded, then the LP is infeasible; it is also possible that both the LP and its dual are infeasible. Duality hence considers three possible scenarios for a maximisation LP P: $\textsc{Opt}(P) = -\infty$, $-\infty < \textsc{Opt}(P) < \infty$, and $\textsc{Opt}(P) = \infty$; then only four scenarios are possible for the primal-dual pair (P, P^*).

Complementation, on the other hand, considers four possible scenarios for P: $\textsc{Opt}(P) = -\infty$, $-\infty < \textsc{Opt}(P) \leq 1$, $1 < \textsc{Opt}(P) < \infty$ and $\textsc{Opt}(P) = \infty$. So this could make up to sixteen scenarios for the primal-complement pair $(P, \overline{P})$. The LP complementation theorem implies that if $1 < \textsc{Opt}(P) < \infty$, then so does $\textsc{Opt}(\overline{P})$ and vice versa. The proof of Theorem 2.1 also shows that if $\textsc{Opt}(P) > 1$, then $\overline{P}$ is feasible, i.e. $\textsc{Opt}(\overline{P}) < \infty$. Therefore, if $\textsc{Opt}(P) = \infty$, then $\textsc{Opt}(\overline{P}) \leq 1$. This leaves nine possible scenarios; for each of those we give an example in Table 3 below.

P	$\overline{P}$	$\textsc{Opt}(P)$	$\textsc{Opt}(\overline{P})$
$\max\{x : x \leq b\}$	$\min\{x : (b-1)x \geq b\}$	$b > 1$	$\frac{b}{b-1} > 1$
$\max\{x : x \leq 0, x \geq 1\}$	$\min\{x : x \leq 0\}$	$-\infty$	$-\infty$
$\max\{-x : x \geq 1, x \leq 0\}$	$\min\{-x : -2x \leq 1, x \leq 0\}$	$-\infty$	0
$\max\{x : x \leq 1, x \geq 2\}$	$\min\{x : 0 \geq 1\}$	$-\infty$	∞
$\max\{-x : x \geq 0\}$	$\min\{-x : x \geq 0\}$	0	$-\infty$
$\max\{x : x = 0\}$	$\min\{x : x = 0\}$	0	0
$\max\{x : x \leq 1\}$	$\min\{x : 0 \geq 1\}$	1	∞
$\max\{x : x \geq 2\}$	$\min\{x : x \leq 2\}$	∞	$-\infty$
$\max\{x : x \geq 0\}$	$\min\{x : x \geq 0\}$	∞	0

Table 3: Examples of the nine possible scenarios for $(\textsc{Opt}(P), \textsc{Opt}(\overline{P}))$. Here x is a single variable.

2.3 Consequence for integer programming

The proof of Theorem 2.1 actually shows that, whenever $\textsc{Opt}(P) > 1$, x is an optimal solution of P if and only if $\frac{1}{\textsc{Opt}(P)-1}x$ is an optimal solution of $\overline{P}$. This has a consequence for integer programming.

For any linear program R, adding the constraint that the variables be integral yields an integer program, which we denote $R^{\mathbb{Z}}$. We consider the LPs in the following form $P : \max\{c^\top x : Ax \leq b\}$ and $Q : \min\{v^\top x : Mx \geq u\}$. For any $s, t \in \mathbb{N}$, we then introduce

$$\begin{aligned} P_s &: \ \max\{c^\top x : Ax \leq sb\}, \\ Q_t &: \ \min\{v^\top x : Mx \geq tu\}. \end{aligned}$$

Clearly, $\textsc{Opt}(P_s) = s\textsc{Opt}(P)$ and $\textsc{Opt}(Q_t) = t\textsc{Opt}(Q)$.

The LP complementation theorem has two consequences for IPs of the form ${P_s}^{\mathbb{Z}}$ or ${Q_t}^{\mathbb{Z}}$. We give these for ${P_s}^{\mathbb{Z}}$ below; their counterparts for ${Q_t}^{\mathbb{Z}}$ are analogous and hence omitted.

Corollary 2.4. *Suppose $P : \max\{c^\top x : Ax \leq b\}$, where A, b, and c are all rational. Let $s, t \in \mathbb{N}$ such that $\tilde{x} \in (\mathbb{Z}/s)^n$ is an optimal solution of P with value $1 + \frac{t}{s} > 1$. Then*

1. *The four optimisation problems P_s, ${P_s}^{\mathbb{Z}}$, $(\overline{P})_t$, and ${(\overline{P})_t}^{\mathbb{Z}}$ all have a common integral optimal solution $\hat{x} = s\tilde{x}$ of value $s + t$.*

2. *We have* $\textsc{Opt}({P_{st}}^{\mathbb{Z}}) = \textsc{Opt}(P_{st})$ *and* $\textsc{Opt}({(\overline{P})_{st}}^{\mathbb{Z}}) = \textsc{Opt}((\overline{P})_{st})$*, thus*

$$\frac{1}{\textsc{Opt}({P_{st}}^{\mathbb{Z}})} + \frac{1}{\textsc{Opt}({(\overline{P})_{st}}^{\mathbb{Z}})} = \frac{1}{st}.$$

Proof. 1. By definition, $\hat{x}$ is an optimal solution of P_s with value $s+t$. Since $\hat{x} = t\frac{1}{\textsc{Opt}(P)-1}\tilde{x}$, we obtain that $\hat{x}$ is also an optimal solution of $(\overline{P})_t$. Moreover, $\hat{x}$ is integral, therefore it is also an optimal solution of ${P_s}^{\mathbb{Z}}$ and ${(\overline{P})_t}^{\mathbb{Z}}$.

2. It is easily seen that for any LP R and any $a \in \mathbb{N}$, if R_a has an integral optimal solution, then so does R_{ab} for any $b \in \mathbb{N}$. By item 1, P_s and $(\overline{P})_t$ both have integral optimal solutions, thus so do P_{st} and $(\overline{P})_{st}$. Applying the LP complementation theorem then finishes the proof.

□

2.4 Bounds

Let $P : \max\{c^\top x : Ax \le b\}$, where $b > 0$ and $A \neq 0$. Let $Q : \min\{v^\top x : Mx \ge u\}$ with $u > 0$ and $M \neq 0$. We remark that $x = 0$ is a feasible solution of both P and Q. Let A_i and M_i denote the i-th rows of A and M, respectively. We define the *rank function* of P and Q, respectively by

$$\rho_P(x) = \max\left\{\frac{A_i x}{b_i} : 1 \le i \le m\right\}$$
$$\sigma_Q(x) = \min\left\{\frac{M_i x}{u_i} : 1 \le i \le m\right\}.$$

Then x is a feasible solution of P (of Q, respectively) if and only if $\rho_P(x) \le 1$ ($\sigma_Q(x) \ge 1$, respectively). We now introduce

$$\alpha(P) = \sup\left\{\frac{c^\top x}{\rho_P(x)} : \rho_P(x) < c^\top x\right\},$$
$$\beta(Q) = \inf\left\{\frac{v^\top x}{\sigma_Q(x)} : 0 < \sigma_Q(x)\right\}.$$

We also introduce the counterparts for the IPs as

$$\alpha(P^{\mathbb{Z}}) = \sup\left\{\frac{c^\top x}{\rho_P(x)} : \rho_P(x) < c^\top x, x \in \mathbb{Z}^n\right\},$$
$$\beta(Q^{\mathbb{Z}}) = \inf\left\{\frac{v^\top x}{\sigma_Q(x)} : 0 < \sigma_Q(x), x \in \mathbb{Z}^n\right\}.$$

Suppose that $1 < \textsc{Opt}(P^{\mathbb{Z}}) < \infty$ and $1 < \textsc{Opt}(\overline{P}^{\mathbb{Z}}) < \infty$. We prove that $\alpha(P^{\mathbb{Z}})$ and $\beta(\overline{P}^{\mathbb{Z}})$ are complement pairs. (The same is true for $\alpha(P)$ and $\beta(\overline{P})$, as we shall prove later.)

Lemma 2.5. *If* $1 < \textsc{Opt}(P^{\mathbb{Z}}) < \infty$ *and* $1 < \textsc{Opt}(\overline{P}^{\mathbb{Z}}) < \infty$*, then we have*

$$\frac{1}{\alpha(P^{\mathbb{Z}})} + \frac{1}{\beta(\overline{P}^{\mathbb{Z}})} = 1.$$

Proof. By definition, we have $\rho_P(x) + \sigma_{\overline{P}}(x) = c^\top x$. Therefore,

$$\begin{aligned} 1 - \frac{1}{\alpha(P^{\mathbb{Z}})} &= 1 - \inf\left\{\frac{\rho_P(x)}{c^\top x} : \rho_P(x) < c^\top x, x \in \mathbb{Z}^n\right\} \\ &= \sup\left\{\frac{c^\top x - \rho_P(x)}{c^\top x} : \rho_P(x) < c^\top x, x \in \mathbb{Z}^n\right\} \\ &= \frac{1}{\inf\left\{\frac{c^\top x}{\sigma_{\overline{P}}(x)} : 0 < \sigma_{\overline{P}}(x), x \in \mathbb{Z}^n\right\}} \\ &= \frac{1}{\beta(\overline{P}^{\mathbb{Z}})}. \end{aligned}$$

□

We obtain the more complete version of Theorem 1.2 as follows.

Theorem 2.6. *Let* $P : \max\{c^\top x : Ax \le b\}$*, where* $b > 0$ *and* $A \neq 0$*. Let* $Q : \min\{v^\top x : Mx \ge u\}$ *with* $u > 0$ *and* $M \neq 0$*.*

1. *If* $\textsc{Opt}(P^{\mathbb{Z}}) > 1$*, then* $1 < \textsc{Opt}(P^{\mathbb{Z}}) \le \alpha(P^{\mathbb{Z}}) \le \alpha(P) = \textsc{Opt}(P)$*.*
2. *If* $\textsc{Opt}(Q^{\mathbb{Z}}) > 1$*, then* $\textsc{Opt}(Q) = \beta(Q) \le \beta(Q^{\mathbb{Z}}) \le \textsc{Opt}(Q^{\mathbb{Z}})$*.*
3. *If* $1 < \textsc{Opt}(P^{\mathbb{Z}}) \le \textsc{Opt}(\overline{P}^{\mathbb{Z}}) < \infty$*, then* $\textsc{Opt}(P) = \alpha(P^{\mathbb{Z}}) \iff \textsc{Opt}(\overline{P}) = \beta(\overline{P}^{\mathbb{Z}})$*.*

Proof. 1. We prove the bounds on α.

(a) $\textsc{Opt}(P^{\mathbb{Z}}) \le \alpha(P^{\mathbb{Z}})$. Let x' be an optimal solution of $P^{\mathbb{Z}}$. Then $0 < \rho_P(x') \le 1 < c^\top x'$, thus

$$\alpha(P^{\mathbb{Z}}) \ge \frac{c^\top x'}{\rho_P(x')} \ge c^\top x' = \textsc{Opt}(P^{\mathbb{Z}}).$$

(b) $\alpha(P^{\mathbb{Z}}) \le \alpha(P)$. By definition.

(c) $\alpha(P) \le \textsc{Opt}(P)$. Let x'' be such that $\alpha(P) = \frac{c^\top x''}{\rho_P(x'')}$, then let $y = \frac{1}{\rho_P(x'')} x''$. We have

$$Ay = \frac{1}{\rho_P(x'')} Ax'' \le \frac{1}{\rho_P(x'')} (\rho_P(x'')b) = b,$$

hence y is a feasible solution of P; its value is $c^\top y = \alpha(P)$.

(d) $\textsc{Opt}(P) \le \alpha(P)$. Same proof as item 1a above.

2. Similar and hence omitted.

3. The pair $(\textsc{Opt}(P), \textsc{Opt}(\overline{P}))$ is a complement pair by the LP complementation theorem, while $(\alpha(P^{\mathbb{Z}}), \beta(\overline{P}^{\mathbb{Z}}))$ is a complement pair by Lemma 2.5. Therefore, $\textsc{Opt}(P) = \alpha(P^{\mathbb{Z}})$ if and only if $\textsc{Opt}(\overline{P}) = \beta(\overline{P}^{\mathbb{Z}})$.

□

3 Fractional hypergraph theory

3.1 Fractional hypergraph parameters

Many important graph parameters, such as the clique number, chromatic number, matching number, etc. can be viewed as the optimal values of IPs defined on hypergraphs related to the original graph. Fractional hypergraph theory then lifts the integrality constraint and focuses on the fractional analogues of those parameters, which are the optimal values of the corresponding LP relaxations. In this section, we review four important fractional hypergraph parameters, and how they are related. A comprehensive account of those parameters can be found in [12].

A (finite) *hypergraph* is a pair $H = (V, E)$, where V is a set of n vertices and E is a multiset of m edges, each being a subset of vertices. Recall the following concepts for a hypergraph H. Its *incidence matrix* is $M = M_H \in \mathbb{R}^{n \times m}$ such that, for all $v \in V$ and $e \in E$,

$$M_{ve} = \begin{cases} 1 & \text{if } v \in e \\ 0 & \text{otherwise.} \end{cases}$$

A vertex is *universal* if it belongs to all edges of H. On the other hand, a vertex is *isolated* if it does not belong to any edge of H. Say an edge e is *complete* if $e = V$ and that it is *empty* if $e = \emptyset$. For a vertex $v \in V$, we denote by E_v the multiset of edges of H that contain v, i.e. $E_v = \{e \in E : v \in e\}$.

We now introduce four LPs related to a hypergraph H; we shall then apply the LP complementation theorem to them. All those LPs have an optimal value in $[1, \infty]$. Technically, if the optimal value is either 1 or ∞, then the LP complementation theorem does not apply. However, we highlight these degenerate cases, which can easily be handled separately. By using the convention that 1 and ∞ form a complement pair, we can then include these degenerate cases in our hypergraph complementation theorem.

A *covering* of H is a set of edges whose union is equal to V. The *covering number* $k(H)$ of H is the minimum size of a covering of H. The *fractional covering number* $k_{\mathrm{f}}(H)$ of H is the optimal value of the following LP, which we give in two forms: a concise matrix form and a more explicit form.

$$\begin{aligned} K(H):\ &\min\{1^\top x : M_H x \geq 1, x \geq 0\} \\ &= \min\left\{\sum_{e\in E} x_e : \sum_{e\in E_v} x_e \geq 1\ \forall v \in V, x_e \geq 0\ \forall e \in E\right\}. \end{aligned}$$

It is easily seen that the covering number is actually the optimal value of $K(H)^{\mathbb{Z}}$. We remark that $K(H)$ is feasible if and only if H has no isolated vertices. Clearly, if $K(H)$ is feasible, then it has an optimal solution. In that case, $k_{\mathrm{f}}(H) = \textsc{Opt}(K(H)) \geq 1$, with strict inequality if and only if H has no complete edges.

A *packing* of H is a set of vertices such that every edge contains at most one of those vertices. The *packing number* $p(H)$ of H is the maximum size of a packing of H. The *fractional packing number* $p_{\mathrm{f}}(H)$ of H is the optimal value of the LP dual to $K(H)$:

$$\begin{aligned} P(H) = K(H)^*:\ &\max\{1^\top y : M_H^\top y \leq 1, y \geq 0\} \\ &= \max\left\{\sum_{v\in V} y_v : \sum_{v\in e} y_v \leq 1\ \forall e \in E, y_v \geq 0\ \forall v \in V\right\}. \end{aligned}$$

Again, the maximum size of a packing of H corresponds to the optimal value of the analogous IP. We remark that $P(H)$ is always feasible. However, $P(H)$ is bounded if and and only if H has no isolated vertices. In that case, $p_{\mathrm{f}}(H) = \textsc{Opt}(P(H)) > 1$ if and only if it has no complete edges. LP duality then yields $p_{\mathrm{f}}(H) = k_{\mathrm{f}}(H)$.

For any hypergraph $H = (V, E)$, its *dual* is $H^* = (E, V^*)$, where $V^* = \{E_v : v \in V\}$. We then have $M_{H^*} = (M_H)^\top$ and $(H^*)^* \cong H$. We note that H has no empty edge if and only if H^* has no isolated vertex, and vice versa.

A *matching* of H is a set of disjoint edges; it corresponds to a packing of H^*. The *fractional matching number* is then $\mu_{\mathrm{f}}(H) = p_{\mathrm{f}}(H^*)$, i.e. the optimal value of:

$$\begin{aligned} M(H) = P(H^*):\ &\max\{1^\top y : M_H y \leq 1, y \geq 0\} \\ &= \max\left\{\sum_{e\in E} y_e : \sum_{e\in E_v} y_e \leq 1\ \forall v \in V, y_e \geq 0\ \forall e \in E\right\}. \end{aligned}$$

A *transversal* of H is a set of vertices such that every edge contains a vertex from that set; it corresponds to a covering of H^*. The *fractional transversal number* is then $\tau_{\mathrm{f}}(H) = k_{\mathrm{f}}(H^*)$, i.e. the optimal value of:

$$\begin{aligned} T(H) = K(H^*):\ &\min\{1^\top x : M_H^\top x \geq 1, x \geq 0\} \\ &= \min\left\{\sum_{v\in V} x_v : \sum_{v\in e} x_v \geq 1\ \forall e \in E, x_v \geq 0\ \forall v \in V\right\}. \end{aligned}$$

Again, LP duality yields $\mu_{\mathrm{f}}(H) = \tau_{\mathrm{f}}(H)$.

Observation 3.1. *In summary, for any H we have $\tau_{\mathrm{f}}(H) = k_{\mathrm{f}}(H^*) = p_{\mathrm{f}}(H^*) = \mu_{\mathrm{f}}(H)$.*

3.2 Hypergraph complementation

We define the *complement* of H as $\overline{H} = (V, \{V \setminus e : e \in E\})$. We then have $M_{\overline{H}} = 1 - M_H$. Hypergraph complementation is an involution that commutes with duality: $\overline{\overline{H}} = H$ and $\overline{(H^*)} = (\overline{H})^*$.

It can be easily shown that

$$\overline{K(H^*)} = K(\overline{H})^*.$$

Therefore, we obtain eight LPs, which are related in Figure 2.

For any $S \subseteq V$, let

$$\rho_H(S) = \max \{|S \cap e| : e \in E\},$$
$$\alpha(H) = \max \left\{ \frac{|S|}{\rho_H(S)} : S \subseteq V, \rho_H(S) > 0 \right\}.$$

We similarly define for any $Z \subseteq E$

$$\sigma_H(Z) = \min \{|\{e \in Z : v \in e\}| : v \in V\},$$
$$\beta(H) = \min \left\{ \frac{|Z|}{\sigma_H(Z)} : Z \subseteq E, \sigma_H(Z) > 0 \right\}.$$

We immediately recognise that $\alpha(H) = \alpha(P(H)^{\mathbb{Z}})$ and $\beta(H) = \beta(K(H)^{\mathbb{Z}})$. Denoting the maximum size of an edge in H as $\epsilon(H) = \max\{|e| : e \in E\}$ and the minimum degree of a vertex in H as $\delta(H) = \min\{|E_v| : v \in V\}$, we have

$$\alpha(H) \geq \frac{|V|}{\epsilon(H)}, \quad \beta(H) \leq \frac{|E|}{\delta(H)}.$$

The next theorem is a more complete version of Theorem 1.3.

Theorem 3.2 (Hypergraph complementation theorem)**.** *For any hypergraph H,*

$$\frac{1}{k_f(H^*)} + \frac{1}{k_f(\overline{H})} = 1.$$

Moreover, we have the bounds

$$p(H) \leq \alpha(H) \leq k_f(H) \leq \beta(H) \leq k(H),$$

with equalities reached as follows:

$$k_f(\overline{H}) = \alpha(\overline{H}) \iff k_f(H^*) = \beta(H^*).$$

Proof. We have $\overline{P(H^*)} = K(\overline{H})$. Theorem 2.1 then shows that $p_f(H^*) = k_f(H^*)$ and $k_f(\overline{H})$ are complement pairs. Theorem 2.6 then gives the other two equations. □

Obviously, the hypergraph complementation theorem holds for all four parameters reviewed in Section 3.1.

Corollary 3.3. *For any hypergraph H,*

$$\frac{1}{k_f(H^*)} + \frac{1}{k_f(\overline{H})} = \frac{1}{p_f(H^*)} + \frac{1}{p_f(\overline{H})} = \frac{1}{\mu_f(H^*)} + \frac{1}{\mu_f(\overline{H})} = \frac{1}{\tau_f(H^*)} + \frac{1}{\tau_f(\overline{H})} = 1.$$

4 Applications to fractional graph theory

4.1 Fractional domination in graphs and digraphs

All the digraphs we consider are simple (no parallel arcs) and irreflexive (no loops). Thus, a *digraph* is a pair $D = (V(D), E(D))$, where $E(D) \subseteq V(D)^2 \setminus \{(v,v) : v \in V(D)\}$. The *adjacency matrix* of D is the $\{0,1\}$-matrix $A_D = (a_{ij} : i,j \in V(D))$, where $a_{ij} = 1$ if and only if $(i,j) \in E(D)$. We define *digraph complement* of D, denoted $\overline{D}$, with $V(\overline{D}) = V(D)$ and $E(\overline{D}) = (V(D)^2 \setminus \{(v,v) : v \in V(D)\}) \setminus E(D)$.

For any $v \in V(D)$, the *open in-neighbourhood* of v is $N_{\mathrm{o}}^{\mathrm{in}}(v) = \{u : (u,v) \in E(D)\}$; the *closed in-neighbourhood* of v is $N_{\mathrm{c}}^{\mathrm{in}}(v) = N_{\mathrm{o}}^{\mathrm{in}}(v) \cup \{v\}$. We thus define two hypergraphs $H_{\mathrm{o}}^{\mathrm{in}}(D)$ and $H_{\mathrm{c}}^{\mathrm{in}}(D)$, both with vertex set $V(D)$, and where the edges of $H_{\mathrm{o}}^{\mathrm{in}}(D)$ are the open in-neighbourhoods of all vertices and the edges of $H_{\mathrm{c}}^{\mathrm{in}}(D)$ are the closed in-neighbourhoods instead. Open and closed out-neighbourhoods are defined similarly, and hence we define $H_{\mathrm{o}}^{\mathrm{out}}(D)$ and $H_{\mathrm{c}}^{\mathrm{out}}(D)$ similarly as well. We note that $M_{H_{\mathrm{o}}^{\mathrm{in}}(D)} = A_D$ and $M_{H_{\mathrm{c}}^{\mathrm{in}}(D)} = I_n + A_D$, where n is the number of vertices in D and I_n is the identity matrix of size n. We then have

$$H_{\mathrm{o}}^{\mathrm{in}}(D)^* \cong H_{\mathrm{o}}^{\mathrm{out}}(D), \quad H_{\mathrm{c}}^{\mathrm{in}}(D)^* \cong H_{\mathrm{c}}^{\mathrm{out}}(D), \quad \overline{H_{\mathrm{o}}^{\mathrm{out}}(D)} = H_{\mathrm{c}}^{\mathrm{out}}(\overline{D}), \quad \overline{H_{\mathrm{c}}^{\mathrm{out}}(D)} = H_{\mathrm{o}}^{\mathrm{out}}(\overline{D}).$$

An *in-dominating set* of D is a set S of vertices such that for any $v \in V(D)$, there exists $s \in S \cap N_{\mathrm{c}}^{\mathrm{in}}(v)$; in other words, it is a transversal of $H_{\mathrm{c}}^{\mathrm{in}}(D)$. Similarly, a *total in-dominating set* of D is a transversal of $H_{\mathrm{o}}^{\mathrm{in}}(D)$. We note that D always has an in-dominating set ($V(D)$ itself), while D has a total in-dominating set if and only if it has no sources (vertices with empty in-neighbourhoods). Out-dominating and total out-dominating sets are defined similarly. See the book by Haynes, Hedetniemi, and Slater for a comprehensive survey of domination problems [7].

The *fractional in-dominating number* of D and the *fractional total out-dominating number* of $\overline{D}$ are then, respectively:

$$\gamma^{\mathrm{in}}{}_{\mathrm{f}}(D) = \tau_{\mathrm{f}}(H_{\mathrm{c}}^{\mathrm{in}}(D)) = \tau_{\mathrm{f}}(H_{\mathrm{c}}^{\mathrm{out}}(D)^*),$$
$$\Gamma^{\mathrm{out}}{}_{\mathrm{f}}(\overline{D}) = \tau_{\mathrm{f}}(H_{\mathrm{o}}^{\mathrm{out}}(\overline{D})) = \tau_{\mathrm{f}}(\overline{H_{\mathrm{c}}^{\mathrm{out}}(D)}).$$

Let us call a vertex v *in-universal* in D if $v \in N_{\mathrm{c}}^{\mathrm{in}}(u)$ for all $u \in V$, i.e. v is a universal vertex of $H_{\mathrm{c}}^{\mathrm{in}}(D)$. We note that $\gamma^{\mathrm{in}}{}_{\mathrm{f}}(D) > 1$ if and only if D has no in-universal vertices; the latter is also equivalent to $\Gamma^{\mathrm{out}}{}_{\mathrm{f}}(\overline{D}) < \infty$. We obtain the following; again the degenerate case of an in-universal vertex is handled by the $(1,\infty)$ complement pair.

Theorem 4.1 (Domination complementation theorem)**.** *For any digraph D,*

$$\frac{1}{\gamma^{\mathrm{in}}{}_{\mathrm{f}}(D)} + \frac{1}{\Gamma^{\mathrm{out}}{}_{\mathrm{f}}(\overline{D})} = 1.$$

We focus on three special cases of Theorem 4.1. Firstly, a *graph* G is a symmetric digraph, i.e. $A_G = A_G^\top$. For a graph G, in-neighbourhoods and out-neighbourhoods coincide. We then refer to $\gamma_{\mathrm{f}}(G) = \gamma^{\mathrm{in}}{}_{\mathrm{f}}(G) = \gamma^{\mathrm{out}}{}_{\mathrm{f}}(G)$ as the fractional dominating number of G; the fractional total dominating number of G is defined and denoted similarly.

Corollary 4.2. *For any graph G,*

$$\frac{1}{\gamma_{\mathrm{f}}(G)} + \frac{1}{\Gamma_{\mathrm{f}}(\overline{G})} = 1.$$

Secondly, a *tournament* T is a digraph where $(i,j) \in E(T)$ if and only if $(j,i) \notin E(T)$. If T is a tournament, then $\overline{T}$ is obtained by reversing the direction of every arc in T. Thus, $H_{\mathrm{o}}^{\mathrm{out}}(\overline{T}) = H_{\mathrm{o}}^{\mathrm{in}}(T)$ and we obtain the following corollary, where the final conclusion follows from Observation 2.2.

Corollary 4.3. *For any tournament T,*

$$\frac{1}{\gamma^{\mathrm{in}}{}_{\mathrm{f}}(T)} + \frac{1}{\Gamma^{\mathrm{in}}{}_{\mathrm{f}}(T)} = 1.$$

In particular, $\gamma^{\mathrm{in}}{}_{\mathrm{f}}(T) \leq 2 \leq \Gamma^{\mathrm{in}}{}_{\mathrm{f}}(T)$.

Thirdly, D is *k-regular* if for every vertex $v \in V(D)$, $|N_{\mathrm{o}}^{\mathrm{in}}(v)| = |N_{\mathrm{o}}^{\mathrm{out}}(v)| = k$. Clearly, if D has n vertices, then D is k-regular if and only if $\overline{D}$ is $(n-1-k)$-regular. The following result is a generalisation of the result in [12, Theorem 7.4.1], which only applies to regular *graphs*.

Corollary 4.4. *For any k-regular digraph D on n vertices,*

$$\gamma^{\mathrm{in}}{}_{\mathrm{f}}(D) = \frac{n}{k+1}, \qquad \Gamma^{\mathrm{out}}{}_{\mathrm{f}}(D) = \frac{n}{k}.$$

Proof. The value $n/(k+1)$ is an obvious upper bound for $\gamma^{\mathrm{in}}{}_{\mathrm{f}}(D)$ (assign $1/(k+1)$ to each vertex); similarly, $n/(n-k-1)$ is an upper bound for $\Gamma^{\mathrm{out}}{}_{\mathrm{f}}(\overline{D})$. By Theorem 4.1, these bounds must be tight. □

4.2 Application to edge toughness of matroids

Let $M = (V, I)$ be a matroid [10], where I is the collection of independent sets of M. A basis of M is a maximal independent set. We then denote the set of bases of M as $B(M)$ and we construct the hypergraph $H_B(M) = (V, B(M))$. The rank function of M is then $\rho_M = \rho_{H_B(M)}$, i.e. $\rho_M(S) = \max\{|S \cap e| : e \in B(M)\}$. The dual matroid $\overline{M}$ is then defined as $H_B(\overline{M}) = \overline{H_B(M)}$, hence its rank function satisfies $\rho_{\overline{M}}(S) = |S| - \rho_M(V) + \rho_M(V \setminus S)$. We note that the dual of a matroid is commonly denoted as M^*, but in this paper, denoting it as $\overline{M}$ better reflects that its definition is in terms of hypergraph complementation, instead of hypergraph duality.

The *edge toughness* (or strength) of M is [12]

$$\sigma'(M) = \min\left\{\frac{|V \setminus S|}{\rho_M(V) - \rho_M(S)} : S \subseteq V, \rho_M(V) > \rho_M(S)\right\}.$$

The edge toughness is well defined unless $\rho_M(V) = 0$. Moreover, $\sigma'(M) = 1$ if and only if M has a coloop, i.e. an element v that belongs to all bases. Say that M is nontrivial if it falls in neither case mentioned above; then its edge toughness satisfies $\sigma'(M) > 1$.

Next, we use the hypergraph complementation theorem to show that the fractional transversal number and fractional matching number of a matroid coincide with its edge toughness.

Theorem 4.5. *For any nontrivial matroid M, we have*

$$\mu_{\mathrm{f}}(H_B(M)) = \tau_{\mathrm{f}}(H_B(M)) = \sigma'(M).$$

The proof of Theorem 4.5 is based on the following lemma. For any hypergraph H, let

$$\gamma(H) = \min\left\{\frac{|T|}{|T| - \rho_H(T)} : T \subseteq V, |T| > \rho_H(T)\right\}.$$

In particular, we can easily check that $\gamma(H_B(\overline{M})) = \sigma'(M)$.

Lemma 4.6. *For any hypergraph H, $\beta(H^*) = \gamma(\overline{H})$.*

Proof. We denote the set of edges of H as E, and the set of edges of $\overline{H}$ as $\overline{E}$. For any $T \subseteq V$, we have

$$\sigma_{H^*}(T) = \min\{|T \cap e| : e \in E\} = |T| - \max\{|T \cap \overline{e}| : \overline{e} \in \overline{E}\} = |T| - \rho_{\overline{H}}(T),$$

and hence

$$\begin{aligned}
\beta(H^*) &= \min\left\{\frac{|T|}{\sigma_{H^*}(T)} : T \subseteq V, \sigma_{H^*}(T) > 0\right\} \\
&= \min\left\{\frac{|T|}{|T| - \rho_{\overline{H}}(T)} : T \subseteq V, |T| > \rho_{\overline{H}}(T)\right\} \\
&= \gamma(\overline{H}).
\end{aligned}$$

□

Proof of Theorem 4.5. Firstly, by the matroid base covering theorem (see [12, Theorem 5.4.1] or [14, Corollary 42.1c]), the fractional covering number of a matroid reaches the α bound in Theorem 3.2. For the dual matroid, we obtain

$$k_{\mathrm{f}}(H_B(\overline{M})) = \alpha(H_B(\overline{M})).$$

Moreover, thanks to Lemma 4.6, we recognise that

$$\sigma'(M) = \gamma(H_B(\overline{M})) = \beta(H_B(M)^*).$$

Applying Observation 3.1 and Theorem 3.2 then yields

$$\mu_{\mathrm{f}}(H_B(M)) = k_{\mathrm{f}}(H_B(M)^*) = \beta(H_B(M)^*) = \sigma'(M).$$

□

Applying the hypergraph complementation theorem, we obtain the following corollary, already given in [12].

Corollary 4.7 (Theorem 5.6.8 in [12])**.** *For any nontrivial matroid, we have*

$$\frac{1}{\sigma'(M)} + \frac{1}{k_{\mathrm{f}}(H_B(\overline{M}))} = 1.$$

In particular, if M_G is the cycle matroid of a graph G, where the elements of M_G are the edges of G and the bases of M_G are all spanning forests of G [10], then the edge toughness of M_G reduces to the edge toughness (a.k.a strength) of G, defined as follows. For any $Z \subseteq E(G)$, let $G - Z$ denote the graph obtained by removing the edges from Z, and let $c(G - Z)$ denote the number of its connected components. Then

$$\sigma'(G) = \min\left\{\frac{|Z|}{c(G-Z) - c(G)} : Z \subseteq E(G), c(G-Z) > c(G)\right\}.$$

We remark that $\sigma'(G)$ is well defined if and only if $E(G)$ is nonempty. Moreover, $\sigma'(G) = 1$ if and only if G has a cut edge, i.e. G has a connected component that is not 2-edge connected.

Denote $H_{SF}(G) = H_B(M_G)$. The matching number of $H_{SF}(G)$ is the maximum number of edge-disjoint spanning forests in G. On the other hand, the transversal number of $H_{SF}(G)$ is the smallest size of an edge cut set of G. In particular, these two quantities are equal to 1 whenever G has a cut edge. When there is no cut edge, by Theorem 4.5, their fractional analogues are equal to the edge toughness of G.

Corollary 4.8. *For any graph G whose connected components are all 2-edge connected,*

$$\mu_{\mathrm{f}}(H_{SF}(G)) = \tau_{\mathrm{f}}(H_{SF}(G)) = \sigma'(G).$$

5 Vertex cover with budget

5.1 The vertex cover hypergraph

Let G be a graph. A vertex cover of G can be defined as a set S of vertices such that $V \setminus S$ is a stable set. We define $H_{VC}(G)$ as the hypergraph whose edges are all the vertex covers of G. Then its complement is $\overline{H_{VC}(G)} = H_{IS}(G)$, whose edges are the stable sets of G. It immediately follows that $k_{\mathrm{f}}(\overline{H_{VC}(G)})$ is equal to $\chi_{\mathrm{f}}(G)$, the *fractional chromatic number* of G. We then denote

$$t_{\mathrm{f}}(G) = \mu_{\mathrm{f}}(H_{VC}(G)),$$

and thanks to Observation 3.1, we have $t_{\mathrm{f}}(G) = k_{\mathrm{f}}(H_{VC}(G)^*)$. We have $\chi_{\mathrm{f}}(G) = 1$ if and only if G is empty, in which case $t_{\mathrm{f}}(G) = \infty$. If G is nonempty, then $\chi_{\mathrm{f}}(G) \geq 2$, with equality if and only if G is bipartite. The hypergraph complementation theorem then yields

$$\frac{1}{t_{\mathrm{f}}(G)} + \frac{1}{\chi_{\mathrm{f}}(G)} = 1.$$

Let us give some properties of the $t_{\mathrm{f}}(G) = \frac{\chi_{\mathrm{f}}(G)}{\chi_{\mathrm{f}}(G)-1}$ quantity.

Bounds Let $\alpha(G)$ denote the independence number of G, $\omega(G)$ denote its clique number and $\chi(G)$ denote its chromatic number. Then the bounds on $\chi_{\mathrm{f}}(G)$ in [12, Chapter 3] and [9], given on the left hand side below, immediately translate to bounds on $t_{\mathrm{f}}(G)$, given on the right hand side below.

$$\begin{aligned}
\chi_{\mathrm{f}}(G) &\geq \frac{n}{\alpha(G)} &\longrightarrow\quad & t_{\mathrm{f}}(G) \leq \frac{n}{n-\alpha(G)},\\
\chi_{\mathrm{f}}(G) &\geq \frac{\chi(G)}{1+\ln\alpha(G)} &\longrightarrow\quad & t_{\mathrm{f}}(G) \leq \frac{\chi(G)}{\chi(G)-1-\ln\alpha(G)},\\
\chi_{\mathrm{f}}(G) &\geq \omega(G) &\longrightarrow\quad & t_{\mathrm{f}}(G) \leq \frac{\omega(G)}{\omega(G)-1},\\
\chi_{\mathrm{f}}(G) &\leq \chi(G) &\longrightarrow\quad & t_{\mathrm{f}}(G) \geq \frac{\chi(G)}{\chi(G)-1}.
\end{aligned}$$

Possible values If G is non-empty, then $t_{\mathrm{f}}(G)$ is a rational number in $(1,2]$. Conversely, for any rational number $q \in (1,2]$, there is G with $t_{\mathrm{f}}(G) = q$ (since $\chi_{\mathrm{f}}(K(n,r)) = n/r$ for the Kneser graph with $n \geq 2r$ (see e.g. [12])).

Complexity Again, complexity results for $\chi_{\mathrm{f}}(G)$ can be converted into complexity results for $t_{\mathrm{f}}(G)$. Thus, for any $1 < s < 2$, determining whether $t_{\mathrm{f}}(G) \geq s$ is NP-complete (an immediate consequence of [6]). On the other hand, $t_{\mathrm{f}}(G)$ can be computed in polynomial time if G is a line graph (see [12, Section 4.5]), or if G is perfect (since the chromatic and fractional chromatic numbers coincide in that case).

5.2 Vertex cover with finite budget

The Vertex Cover with Budget (VCB) problem is defined as follows. Let G be a graph and b a positive integer. For any family of t vertex covers $S = \{S_1, \dots, S_t\}$ of G, we refer to the budget of S as the maximum number of times a particular vertex appears in S:

$$\max\{|\{i : v \in S_i\}| : v \in V\}.$$

For any $b \geq 1$, we denote the cardinality of the largest family of vertex covers with budget at most b as $t_b(G)$. The problem is, given G and b, to determine $t_b(G)$.

We note that VCB differs from the so-called Budgeted Maximum Vertex Cover Problem (see [2] and references therein).

A *b-fold matching* of a hypergraph H is a set of edges of H such that every vertex is contained in at most b edges (so that a matching is a 1-fold matching). The maximum size of a b-fold matching is denoted as $\mu_b(H)$. We immediately obtain that $t_b(G) = \mu_b(H_{VC}(G))$. Similarly, a *c-fold covering* is a set of edges of H such that every vertex is contained in at least c edges. The smallest size of a c-fold covering of H is denoted as $k_c(H)$. We then have [12, Theorem 1.2.1]

$$\begin{aligned}
\mu_{\mathrm{f}}(H) &= \lim_{b\to\infty} \frac{\mu_b(H)}{b} = \max_{b\in\mathbb{N}} \frac{\mu_b(H)}{b},\\
k_{\mathrm{f}}(H) &= \lim_{c\to\infty} \frac{k_c(H)}{c} = \max_{c\in\mathbb{N}} \frac{k_c(H)}{c}.
\end{aligned}$$

Moreover, there exist β and γ such that $\mu_{l\beta} = l\beta\mu_{\mathrm{f}}(H)$ and $k_{l\gamma} = l\gamma k_{\mathrm{f}}(H)$ for all $l \in \mathbb{N}$. Therefore, $t_{\mathrm{f}}(G)$ is the limit of the time-per-budget ratio $t_b(G)/b$.

Proposition 5.1. *For any G,*

$$t_{\mathrm{f}}(G) = \lim_{b\to\infty} \frac{t_b(G)}{b} = \max_{b\to\infty} \frac{t_b(G)}{b}.$$

Moreover, there exists $\beta \in \mathbb{N}$ such that $t_{l\beta}(G) = t_{\mathrm{f}}(G) \cdot l\beta$ for all $l \in \mathbb{N}$.

We now obtain more precise results about $t_b(G)$.

Proposition 5.2. *For any G and any b, we have*

$$\left\lfloor \frac{\chi(G)}{\chi(G)-1} \cdot b \right\rfloor \leq t_b(G) \leq \left\lfloor \frac{\omega(G)}{\omega(G)-1} \cdot b \right\rfloor$$

Proof. If there is a homomorphism from G' to G, which we denote as $G' \to G$, then $t_b(G) \leq t_b(G')$. Since $K_{\omega(G)} \to G \to K_{\chi(G)}$, we obtain $t_b(K_{\chi(G)}) \leq t_b(G) \leq t_b(K_{\omega(G)})$. It is then easy to verify that $t_b(K_n) = \left\lfloor \frac{n}{n-1} \cdot b \right\rfloor$ for all $n \geq 1$. Hence the result. □

Since the chromatic number of a perfect graph can be computed in polynomial time [6], we obtain the following

Corollary 5.3. *If G is a perfect graph, then for any b, $t_b(G) = \left\lfloor \frac{\chi(G)}{\chi(G)-1} \cdot b \right\rfloor$ can be computed in polynomial time.*

The highest time $t_b(G)$ is only achieved for bipartite graphs, as seen below.

Proposition 5.4. *The following are equivalent.*

(a) $t_{\mathrm{f}}(G) = 2$.

(b) $t_b(G) = 2b$ *for some* $b \geq 1$.

(c) $t_b(G) = 2b$ *for all* $b \geq 1$.

(d) G *is bipartite.*

Proof. We have (d) $\implies$ (c) $\implies$ (b) $\implies$ (a). Conversely, $t_{\mathrm{f}}(G) = 2$ if and only if $\chi_{\mathrm{f}}(G) = 2$, which in turn is equivalent to G being bipartite. □

We obtain a final result on the computational complexity of decision problems related to $t_b(G)$.

Theorem 5.5. *For any $b, c \geq 1$ and any hypergraph H,*

$$\mu_b(H) \geq b + c \iff k_c(\overline{H}) \leq b + c.$$

Proof. It is easy to verify that each statement is equivalent to the next, in the following sequence:

- $\mu_b(H) \geq b + c$.
- There exist $b + c$ edges of H, say $e_1, \dots, e_{b+c}$, such that for any $v \in V$, $|\{i : v \in e_i\}| \leq b$.
- There exist $b + c$ edges of $\overline{H}$, say $f_1, \dots, f_{b+c}$, such that for any $v \in V$, $|\{i : v \in f_i\}| \geq c$.
- $k_c(H) \leq b + c$.

□

A *c-multicolouring* of a graph G is a colouring of its vertices, such that each vertex is assigned a set of c distinct colours, and where the sets of colours of any two adjacent vertices are disjoint [3]. For $H = H_{VC}(G)$, we have $\mu_b(H) = t_b(G)$ and $k_c(\overline{H}) = \chi_c(G)$, the smallest number of colours in a c-multicolouring of G. It follows from Theorem 5.5 that for any $b \geq 1$ and $c \geq 1$, $t_b(G) \geq b + c$ if and only if $\chi_c(G) \leq b + c$. For $c = b$, as proved in Proposition 5.4, deciding whether $t_b(G) = 2b$ can be done in polynomial time. On the other hand, since for any c and any $a > 2c$, deciding whether a graph G satisfies $\chi_c(G) \leq a$ is NP-complete (see [12, Section 3.9]), we obtain the following corollary.

Corollary 5.6. *For any $b \geq 2$ and any $1 \leq c \leq b-1$, it is NP-complete to decide whether $t_b(G) \geq b+c$.*

6 Conclusion and future work

In this paper, we introduced LP complementation, and established the LP complementation theorem. We then illustrated the potential applications of LP complementation to Integer Linear Programming, graph and hypergraph theory, and algorithmic problems on graphs.

We believe that LP complementation has an interesting potential that needs to be uncovered. We can highlight several ways in which LP complementation could be applied further:

- LP and hypergraph complementation for structural hypergraph theory. The hypergraph complementation operation is very natural, and yet it does not seem to have been studied yet.
- Application of LP complementation in graph theory and combinatorics. Any 0/1 matrix can be interpreted as the bipartite adjacency matrix of a bipartite graph G. The complementation operation then corresponds to the bipartite adjacency matrix of the bipartite complement of G. Can LP complementation be used to establish relationships between the properties of a bipartite graph and its bipartite complement?
- Further applications to combinatorial problems "with budget." Including the budget b and considering the limit of a particular quantity as b tends to infinity naturally involves optimal values of LPs.
- Algorithmic applications of LP complementation. It is well-known that LP duality can be used to create efficient primal-dual algorithms for LP. Can LP complementation be used in a similar way?

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