Finding Secluded Places of Special Interest in Graphs*

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Abstract

Finding a vertex subset in a graph that satisfies a certain property is one of the most-studied topics in algorithmic graph theory. The focus herein is often on minimizing or maximizing the size of the solution, that is, the size of the desired vertex set. In several applications, however, we also want to limit the "exposure" of the solution to the rest of the graph. This is the case, for example, when the solution represents persons that ought to deal with sensitive information or a segregated community. In this work, we thus explore the (parameterized) complexity of finding such *secluded* vertex subsets for a wide variety of properties that they shall fulfill. More precisely, we study the constraint that the (open or closed) neighborhood of the solution shall be bounded by a parameter and the influence of this constraint on the complexity of minimizing separators, feedback vertex sets, $\mathcal{F}$-free vertex deletion sets, dominating sets, and the maximization of independent sets.

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1 Introduction

In many optimization problems on graphs, one searches for a minimum or maximum cardinality subset of vertices and edges satisfying certain properties, like a minimum s-t path, a maximum independent set, or a minimum dominating set. In several applications, however, it is important to also limit the *exposure* of the solution [5, 15]. For instance, we may want to find a way to send sensitive information that we want to protect from a vertex s to a vertex t in a network. If we assume that the information is exposed to all vertices on the way and

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all of their neighbors, limiting the exposure means to find an s-t path with a small closed neighborhood [5]. Another example is the search for segragated communities in social networks [15]. Herein, we search for dense subgraphs which are exposed to few neighbors in the rest of the graph. In addition to being a natural constraint in these applications, restricting the exposure of the solution may also yield more efficient algorithms [15, 13, 16, 14].

In accordance with previous work, we call a solution *secluded* if it has a small exposure [5]. Secluded paths and Steiner trees have been studied before [5, 10]. Our aim in this paper is to study the constraint of being secluded on the complexity of diverse vertex-subset optimization problems.

Inspired by Chechik et al. [5], we first measure the exposure of a solution S by the size of the closed neighborhood $N_G[S] = S \cup \bigcup_{v\in S} N_G(v)$ of S in the input graph G. Given a predicate $\Pi(G, S)$ that determines whether S is a solution for input graph G, we hence study the following problem.

Secluded Π
Input: A graph $G = (V, E)$ and an integer k.
Question: Is there a subset $S \subseteq V$ of vertices such that S satisfies $\Pi(G, S)$ and $|N_G[S]| \le k$?

It makes sense to also control the size of the solution and its neighborhood in the graph directly. For example, when sending sensitive information from s to t as above, we may simultaneously aim to optimize latency, that is, minimize the number of vertices in the communication path and limit the exposure. Hence, our second measure of exposure of the solution is the size of the open neighborhood $N_G(S) = (\bigcup_{v\in S} N_G(S)) \setminus S$. This leads to the following problem formulation.

Small (Large) Secluded Π
Input: A graph $G = (V, E)$ and two integers k, ℓ.
Question: Is there a subset $S \subseteq V$ of vertices of G such that S satisfies $\Pi(G, S)$, $|S| \le k$, and $|N_G(S)| \le \ell$ (resp. $|S| \ge k$, and $|N_G(S)| \le \ell$)?

We study both problems in the framework of parameterized complexity. As a parameter for Secluded Π we use the size k of the closed neighborhood and as parameters for Small Secluded Π we use the size k of the solution as well as the size ℓ of the open neighborhood.

The predicates $\Pi(G, S)$ that we study are s-t Separator, Feedback Vertex Set (FVS), $\mathcal{F}$-free Vertex Deletion ($\mathcal{F}$-FVD) (for an arbitrary finite family $\mathcal{F}$ of graphs), and Independent Set (IS). Perhaps surprisingly, we find that Secluded s-t Separator is polynomial-time solvable, whereas Small Secluded s-t Separator becomes NP-complete. The remaining problems are NP-complete. For them, roughly speaking, we prove that fixed-parameter tractability results for Π parameterized by the solution size carry over to Secluded Π parameterized by k. For Small Secluded Π parameterized by ℓ, however, we mostly obtain W[1]-hardness. On the positive side, for Small Secluded $\mathcal{F}$-FVD we prove fixed-parameter tractability when parameterized by $k + \ell$.

We also study, for two integers $p < q$, the p-secluded version of q-Dominating Set (q-DS): a vertex set S is a *q-dominating set* if every vertex of $V \setminus S$ has distance at most q to some vertex in S. Herein, by p-secluded we mean that we upper bound the size of the distance-p-neighborhood of the solution S. Interestingly, this problem admits a complexity dichotomy: Whenever $2p > q$, (Small) p-Secluded q-Dominating Set is fixed-parameter tractable with respect to k (with respect to $k + \ell$), but it is W[2]-hard otherwise.

We also study polynomial-size problem kernels for our secluded problems. Here we observe that the polynomial-size problem kernels for Feedback Vertex Set and $\mathcal{F}$-free Vertex

	Problem	Complexity	Parameterized Compl.	Kernelization
Secluded	s-t separator	P (Thm. 2.1)	-	-
	q-DS, $2p \leq q$	NP-h. (Thm. 3.2)	W[2]-h. wrt. k (Thm. 3.3)	-
	q-DS, $2p > q$		FPT wrt. k (Cor. 3.6)	no PK wrt. k (Thm. 3.2)
	$\mathcal{F}$-free VD	NP-h. (Thm. 4.1)	FPT wrt. k (Thm. 4.2)	PK wrt. k (Thm. 4.2)
	FVS	NP-h. (Thm. 5.1)	FPT wrt. k (Thm. 5.2)	PK wrt. k (Thm. 5.2)
Small Secluded	s-t separator	NP-h. (Thm. 2.2)	W[1]-h. wrt. ℓ (Thm. 2.2)	no PK wrt. $k + \ell$ (Thm. 2.3)
	q-DS, $2p \leq q$	NP-h.*	W[2]-h. wrt. $k+\ell$ (Cor. 3.4)	-
	q-DS, $2p > q$		FPT wrt. $k+\ell$ (Thm. 3.5)	no PK wrt. $k + \ell$ (Cor. 3.4)
	$\mathcal{F}$-free VD	NP-h.*	FPT wrt. $k+\ell$ (Thm. 4.7)	?
	FVS	NP-h.*	W[1]-h. wrt. ℓ (Thm. 5.3)	?
	Large Secl. IS	NP-h.*	W[1]-h. wrt. $k+\ell$ (Thm. 6.1)	-

Table 1 Overview of our results. PK stands for polynomial kernel. The results marked by an asterisk follow by a straightforward reduction from the non-secluded variant.

Deletion carry over to their Secluded variants, but otherwise we obtain mostly absence of polynomial-size problem kernels unless the polynomial hierarchy collapses.

A summary of our results is given in Table 1.

Related work. Secluded Path and Secluded Steiner Tree were introduced and proved NP-complete by Chechik et al. [5]. They obtained approximation algorithms for both problems with approximation factors related to the maximum degree. They also showed that Secluded Path is fixed-parameter tractable with respect to the maximum vertex degree of the input graph, whereas vertex weights lead to NP-hardness for maximum degree four.

Fomin et al. [10] studied the parameterized complexity of Secluded Path and Secluded Steiner Tree, showing that both are fixed-parameter tractable even in the vertex-weighted setting. Furthermore, they showed that Secluded Steiner Tree is fixed-parameter tractable with respect to $r+p$, where $r = k-s$, k is the desired size of the closed neighborhood of the solution, s is the size of an optimum Steiner tree, and p is the number of terminals. On the other hand this problem is co-W[1]-hard when parameterized by r only.

The concept of isolation was thoroughly explored for finding dense subgraphs [15, 13, 16, 14]. Herein, chiefly the constraint that the vertices in the solution shall have maximum/minimum/average outdegree bounded by a parameter was considered [15, 13, 16], leading to various parameterized tractability and hardness results. Also the overall number of edges outgoing has been studied recently [14].

Preliminary observations. Concerning classical computational complexity, the Small (Large) Secluded variant of a problem is at least as hard as the nonsecluded problem, by a simple reduction in which we set $\ell = n$, where n denotes the number of vertices in the graph. Since this reduction is a parameterized reduction with respect to k, parameterized hardness results for this parameter transfer, too. Furthermore, observe that hardness also transfers from Secluded Π to Small Secluded Π for all problems Π, since Secluded Π allows for a parameterized Turing reduction to Small Secluded Π: try out all k' and ℓ' with $k = k' + \ell'$. Additionally, many tractability results (in particular polynomial time solvability and fixed-parameter tractability) transfer from SMALL SECLUDED Π parameterized by $(k+\ell)$ to SECLUDED Π parameterized by k.

▶ **Observation 1.1.** SECLUDED Π parameterized by k is parameterized Turing reducible to SMALL SECLUDED Π parameterized by $(k+\ell)$ for all predicates Π.

Therefore, for the SMALL (LARGE) SECLUDED variants of the problems the interesting cases are those where Π itself is easy, or when ℓ is a parameter.

Notation. We use standard notation from parameterized complexity and graph theory. All graphs in this paper are undirected. We denote $d_G(u,v)$ the *distance* between vertices u and v in G, i.e., the length of a shortest u-v path in G. For a set V' of vertices and a vertex $v \in V$ we let the distance of v from V' denoted $d_G(v,V')$ be equal to $\min\{d_G(u,v) \mid u \in V'\}$. We denote $N_G^d[V'] = \{v \mid d_G(v,V') \leq d\}$ and $N_G^d(V') = N_G^d[V'] \setminus V'$ for any $d \geq 0$ (hence $N_G^0(V') = \emptyset$). We omit the index if the graph is clear from context and also use $N[V']$ for $N^1[V']$ and similarly for $N(V')$. If $V' = \{v\}$, then we write $N^d[v]$ in place of $N^d[\{v\}]$, etc.

Organization. The following sections each treat the secluded versions of one problem. The association is Section 2: s-t SEPARATOR, Section 3: q-DOMINATING SET, Section 4: $\mathcal{F}$-FREE VERTEX DELETION, Section 5: FEEDBACK VERTEX SET, and Section 6: INDEPENDENT SET. We give a brief summary and directions for future research in Section 7. Due to space constraints, the proofs of theorems and lemmata marked with $(\star)$ and parts of other proofs are deferred to an appendix.

2 s-t Separator

In this section we study the problem of finding small s-t separator from the secluded perspective. We show that SECLUDED s-t SEPARATOR is in P, while SMALL SECLUDED s-t SEPARATOR is NP-hard and $W[1]$-hard with respect to the size of the open neighborhood. Moreover, we also exclude the existence of polynomial kernel for the latter problem with respect to the sum of the bounds.

2.1 Secluded s-t Separator

In this subsection we show that the following problem can be solved in polynomial time.

SECLUDED s-t SEPARATOR
Input: A graph $G=(V,E)$, two distinct vertices $s,t \in V$, and an integer k.
Question: Is there an s-t separator $S \subseteq V \setminus \{s,t\}$ such that $|N_G[S]| \leq k$?

▶ **Theorem 2.1.** SECLUDED s-t SEPARATOR *can be solved in polynomial time.*

Proof. We reduce the problem to an ordinary s-t separator in an auxiliary graph. Let $(G=(V,E),s,t,k)$ be the input instance and G'' be a graph obtained from G by adding two vertices s' and t' and making s' only adjacent to s and t' only adjacent to t'. Now let $G'=(V',E')$ be the graph obtained as a third power of G'', i.e., $V' = V(G') = V(G'') = V \cup \{s',t'\}$ and $\{u,v\} \in E'$ if and only if $d_{G''}(u,v) \leq 3$.

We claim that there is an s-t separator S in G with $|N[S]| \leq k$, if and only if there is an s'-t' separator S' in G' with $S' \leq k$. The theorem then follows as we can construct G' and find the minimum s'-t' separator in G' in polynomial time using standard methods, e.g., based on network flows.

For one direction, let S be an s-t separator in G with $|N[S]| \leq k$. Observe that S then also constitutes an s'-t' separator in G'' as every path in G'' from s' must go through s and every path to t' must go through t. We claim that $S' = N[S]$ is an s'-t' separator in G'. Suppose for contradiction that there is an s'-t' path $P = p_0, p_1, \ldots, p_q$ in $G' - S'$. Let A' be the set of vertices of the connected component of $G'' - S$ containing s' and let a be the last index such that $p_a \in A'$ (note that $p_0 = s' \in A'$ and $p_q = t' \notin A'$ by definition). It follows that $p_{a+1} \notin A'$ and, since $\{p_a, p_{a+1}\} \in E'$, there is a p_a-p_{a+1} path P' in G'' of length at most 3. As we have $p_a \in A'$ and $p_{a+1} \in V \setminus (A' \cup S')$ and $G[A']$ is a connected component of $G'' - S$, there must be a vertex x of S on P'. Since neither p_a nor p_{a+1} is in $S' = N[S]$, it follows that $d_G(p_a, x) \geq 2$ and $d_G(p_{a+1}, x) \geq 2$. This contradicts P' having length at most 3.

For the other direction, let S' be an s'-t' separator in G' of size at most k. Let A' be the vertex set of the connected component of $G' - S'$ containing s'. Consider the set $S = \{v \in S' \mid d_{G''}(v, A') = 2\}$. We claim that S is an s-t separator in G and, moreover, $N[S] \subseteq S'$ and, hence, $|N[S]| \leq k$. As to the second part, we have $S \subseteq S'$ by definition. Suppose for contradiction that there was a vertex $u \in N(S) \setminus S'$ that is a neighbor of $v \in S$. Then, since $d_{G''}(v, A') = 2$, we have $d_{G''}(u, A') \leq 3$, u has a neighbor in A' in G', and, thus u is in A'. This implies that $d_{G''}(v, A') = 1$, a contradiction.

Concerning the first part, we prove that S is an s'-t' separator in G''. Since it contains neither s nor t, it follows that it must be also an s-t separator in G. Assume for contradiction that there is an s'-t' path in $G'' - S$. This implies that $d_{(G''-S)}(t', A')$ is well defined (and finite). Let $q := d_{(G''-S)}(t', A')$ and P be the corresponding shortest path in $G'' - S$. Let us denote $P = p_0, \ldots, p_q$ with $p_q = t'$ and $p_0 \in A'$. If $d_{G''}(t', A') \leq 3$, then t' has a neighbor in A' in G', and therefore it is in A' contradicting our assumption that S' is an s'-t' separator in G'. As $t' = p_q$, we have $q \geq 3$. Since $d_{G''}(p_0, A') = 0$, $d_{G''}(p_q, A') > 3$, and $d_{G''}(p_{i+1}, A') \leq d_{G''}(p_i, A') + 1$ for every $i \in \{0, \ldots q-1\}$, there is an a such that $d_{G''}(p_a, A') = 2$. If p_a is not in S', then p_a is in A', contradicting our assumptions on P and q as $a \geq 2$. Therefore we have $d_{G''}(p_a, A') = 2$ and p_a is in S'. It follows that p_a is in S — a contradiction. ◀

2.2 Small Secluded s-t Separator

In this subsection we consider the following problem.

Small Secluded s-t Separator
Input: A graph $G = (V, E)$, two distinct vertices $s, t \in V$, and two integers k, ℓ.
Question: Is there an s-t separator $S \subseteq V \setminus \{s, t\}$ such that $|S| \leq k$ and $|N_G(S)| \leq \ell$?

We show that, in contrast to Secluded s-t Separator, the above problem is NP-hard. Moreover, at the same time, we show parameterized hardness with respect to ℓ.

▶ **Theorem 2.2.** Small Secluded s-t Separator *is* NP*-hard and* $W[1]$*-hard when parameterized by* ℓ*.*

Proof. We provide a parameterized reduction from Clique parameterized by k. Let $(G = (V, E), k)$ be an instance of Clique and assume that $k > 1$ and G does not contain isolated vertices. We construct an equivalent instance (G', s, t, k', ℓ') of Small Secluded s-t Separator as follows. (See **??** in Appendix). To obtain G' from G, first subdivide each edge, denote the new vertex on edge $\{a, b\} \in E$ as x^{ab} and let $X := \{x^{ab} \mid \{a, b\} \in E\}$. Then add a vertex x and make it adjacent to all vertices in V, that is, to all vertices of the original graph. Finally, add two vertices s and t and make both of them adjacent only to x. To finish the construction, set $k' = |V(G')| - \binom{k}{2} - k - 2$ and $\ell' = k + 2$. In words, the task is to find an s-t separator in G' that contains all but at least $\binom{k}{2} + k + 2$ vertices. We defer the proof of the equivalence of the instances to Subsection **??** of the appendix. ◀

It is an interesting open question, whether Small Secluded s-t Separator is FPT parameterized by $k+\ell$ or solely by k. We conjecture that at least the former is the case. Nevertheless, we show that, under standard assumptions, the problem does not admit a polynomial kernel with respect to any of these parameterizations.

▶ **Theorem 2.3** ($\star$). *Unless* NP $\subseteq$ coNP / poly, Small Secluded s-t Separator *parameterized by* $k+\ell$ *does not admit a polynomial kernel.*

3 q-Dominating Set

In this section, for two constants $p, q \in \mathbb{N}$ with $0 \leq p < q$, we consider the following problems:

p-Secluded q-Dominating Set
Input: A graph $G = (V, E)$, and integer k.
Question: Is there $S \subseteq V$ such that $V = N_G^q[S]$, and $|N_G^p[S]| \leq k$?

Small p-Secluded q-Dominating Set
Input: A graph $G = (V, E)$, and two integers k, ℓ.
Question: Is there $S \subseteq V$ such that $V = N_G^q[S]$, $|S| \leq k$, and $|N_G^p(S)| \leq \ell$?

For $p = 0$ the size restrictions in both cases boil down to $|S| \leq k$. This is the well known case of q-Dominating Set (also known as q-Center) which is NP-hard and W[2]-hard with respect to k (see, e.g., [18]). Therefore, for the rest of the section we focus on the case $p > 0$. Additionally, by a simple reduction from q-Dominating Set letting $\ell = |V(G)|$, we arrive at the following observation.

▶ **Observation 3.1.** For any $0 < p < q$, Small p-Secluded q-Dominating Set is W[2]-hard with respect to k.

Further note that, by Observation 1.1, hardness results for p-Secluded q-Dominating Set with respect to parameter k transfer to Small p-Secluded q-Dominating Set with respect to parameter $(k + \ell)$. Hence, we start with showing NP-hardness and W[2]-hardness with respect to k for p-Secluded q-Dominating Set.

For the hardness results, we reduce from the following problem:

Set Cover
Input: A finite universe U, a family $F \subseteq 2^U$, and an integer k.
Question: Is there a subset $X \subseteq F$ such that $|X| \leq k$ and $\bigcup_{x \in X} x = U$?

We write $\bigcup X$ short for $\bigcup_{x \in X} x$. It is known that Set Cover is NP-complete, W[2]-hard with respect to k, and admits no polynomial kernel with respect to $|F|$, unless NP $\subseteq$ coNP / poly [6].

▶ **Theorem 3.2.** *For any* $0 < p < q$, p-Secluded q-Dominating Set *is* NP*-hard. Moreover, it does not admit a polynomial kernel with respect to* k, *unless* NP $\subseteq$ coNP / poly.

Proof. We give a polynomial parameter transformation from Set Cover parameterized by $|F|$. Let (U, F, k) be an instance of Set Cover. Without loss of generality we assume that $0 \leq k < |F|$.

Construction. Let $k' = p + 1 + |F| \cdot p + k$. We construct graph G as follows. We start the construction by taking two vertices s and r and three vertex sets $V_U = \{u \mid u \in U\}$, $V_F = \{v_A \mid A \in F\}$, and $V_F' = \{v_A' \mid A \in F\}$. We connect vertex r with vertex s by a path of length exactly q. For each $A \in F$ we connect vertices v_A and r by an edge and vertices v_A and v'_A by a path $t_0^A, t_1^A, \ldots, t_p^A$ of length exactly p, where $t_0^A = v_A$ and $t_p^A = v'_A$. Let us denote T the set of vertices on all these paths (excluding the endpoints). All introduced paths are internally disjoint and the internal vertices are all new. We connect a vertex $v'_A \in V'_F$ with a vertex $u \in V_U$ by an edge if and only if $u \in A$. Furthermore, we introduce a clique C_U of size k' and make all its vertices adjacent to each vertex in $V'_F \cup V_U$.

For each $u \in U$ we create a path $b_0^u, b_1^u, \ldots, b_{q-p-2}^u$ of length exactly $q-p-2$ such that $b_0^u = u$ and the other vertices are new. Let us denote the set of all new vertices introduced in this step B. Furthermore, in this case, for each $h \in \{0, \ldots, q-p-2\}$ we introduce a clique C_h^u of size k' and make all its vertices adjacent to vertex b_h^u. Let us denote the set of all vertices introduced in this step C. If $q - p = 1$ we do not introduce any new vertices.

We defer the proof of the equivalence of the original and constructed instance to Subsection **??** of appendix. ◀

In the following, we observe that the parameterized complexity of both problems varies for different choices of p and q.

▶ **Theorem 3.3** ($\star$). *For any $0 < p \leq \frac{1}{2}q$, p-Secluded q-Dominating Set is* W[2]*-hard with respect to k.*

By Observation 1.1, these hardness results for p-Secluded q-Dominating Set imply hardness for Small p-Secluded q-Dominating Set, hence we get the following corollary.

▶ **Corollary 3.4.** *For any $0 < p < q$, Small p-Secluded q-Dominating Set is* NP*-hard. Moreover, it does not admit a polynomial kernel with respect to $(k + \ell)$, unless* NP $\subseteq$ coNP / poly*. For any $0 < p \leq \frac{1}{2}q$, Small p-Secluded q-Dominating Set is* W[2]*-hard with respect to $(k + \ell)$.*

Now we look at the remaining choices for p and q, that is all p, q with $p > \frac{1}{2}q$. In these cases we can show fixed-parameter tractability.

▶ **Theorem 3.5.** *For any $p > \frac{1}{2}q$, Small p-Secluded q-Dominating Set can be solved in $O(mk^{k+1}(k + \ell)^{qk+1})$ time and, hence, it is fixed-parameter tractable with respect to $k + \ell$.*

Proof. Consider an instance (G, k, ℓ) of Small p-Secluded q-Dominating Set and a solution S for that instance. If $x \in S$, then $|N^p[x]| \leq k + \ell$, since $|S| \leq k$ and $|N^p(S)| \leq \ell$. Moreover, $|N^p[x]| \leq k + \ell$ implies $|N[v]| \leq k + \ell$ for every $v \in N^{p-1}[x]$. It follows that, if $|N^p[y]| \leq k + \ell$ and $y \notin S$, then for each $x \in N^q[y] \cap S$ every vertex on every x-y path of length at most $2p - 1 \geq q$ must be of degree at most $k + \ell - 1$, since each such vertex is in distance at most $p - 1$ to x or y.

If $k + \ell = 1$, then either G has at most one vertex or (G, k, ℓ) is a no-instance. Hence we further assume that $k + \ell \geq 2$. We call vertices u and v *linked*, if there is a path of length at most q between u and v in G such that the degree of every vertex on the path is at most $k + \ell - 1$. Let $B[u] = \{v \mid u \text{ and } v \text{ are linked}\}$. We claim that for any v, we have $|B[v]| \leq (k + \ell)^q$. We defer the proof of this claim to Subsection **??** of appendix.

Let $Y = \{y \mid |N^p[y]| \leq k + \ell\}$. Obviously we have $S \subseteq Y$, since $|N^p[S]| \leq k + \ell$. If $y \in Y \setminus S$, then there is $x \in S$ such that x and y are linked. It follows that $y \in B[x]$ and $Y \subseteq \bigcup_{x \in S} B[x]$. Thus $|Y| \leq k \cdot (k + \ell)^q \leq (k + \ell)^{q+1}$.

This suggest the following algorithm for Small p-Secluded q-Dominating Set. Find the set Y. If $|Y| > k \cdot (k + \ell)^q$, then answer NO. Otherwise for each $k' \leq k$ and each size-k' subset S of Y check, whether S is a p-secluded q-dominating set in G. If any of the sets succeeds, then return this set, otherwise answer NO. Since $S \subseteq Y$, this check is exhaustive.

As to the running time, the set Y can be determined in $O(n(k+\ell))$ time by running a BFS from each vertex and stopping it after it discovers $k+\ell$ vertices or all vertices in distance at most p, whichever occurs earlier. Then, there are $k \cdot \binom{k\cdot(k+\ell)^q}{k} \leq k^{k+1}(k+\ell)^{qk}$ candidate subsets of Y. For each such set S we can check whether it is a p-secluded q-dominating set in G by running a BFS from each vertex of S and marking the vertices which are in distance at most p and at most q, respectively. This takes $O(m(k+\ell))$ time. Hence, in total, the algorithm runs in $O(mk^{k+1}(k+\ell)^{qk+1})$ time. ◀

By Observation 1.1, the previous result transfers to p-Secluded q-Dominating Set parameterized by k.

▶ **Corollary 3.6.** *For any $p > \frac{1}{2}q$, p-*Secluded q-Dominating Set *is fixed-parameter tractable with respect to k.*

4 $\mathcal{F}$-free Vertex Deletion

In this section, we study the $\mathcal{F}$-free Vertex Deletion ($\mathcal{F}$-FVD) problem for families $\mathcal{F}$ of graphs with at most a constant number c of vertices, that is, the problem of destroying all induced subgraphs isomorphic to graphs in $\mathcal{F}$ by at most k vertex deletions.

4.1 Secluded $\mathcal{F}$-free Vertex Deletion

In this section, we prove a polynomial-size problem kernel for Secluded $\mathcal{F}$-free Vertex Deletion, where $\mathcal{F}$ is a family of graphs with at most a constant number c of vertices:

Secluded $\mathcal{F}$-free Vertex Deletion
Input: A graph $G = (V, E)$, and an integer k.
Question: Is there a set $S \subseteq V$ such that $G - S$ is $\mathcal{F}$-free and $|N_G[S]| \leq k$?

Henceforth, we call a set $S \subseteq V$ such that $G - S$ is $\mathcal{F}$-free an *$\mathcal{F}$-free vertex deletion set.*

Note that Secluded $\mathcal{F}$-free Vertex Deletion can be polynomial-time solvable for some families $\mathcal{F}$ for which $\mathcal{F}$-FVD is NP-hard: Vertex Cover (where $\mathcal{F}$ contains only the graph consisting of a single edge) is NP-hard, yet any vertex cover S satisfies $N[S] = V$. Therefore, an instance to Secluded Vertex Cover is a yes-instance if and only if $k \geq n$. In general, however, Secluded $\mathcal{F}$-Free Vertex Deletion is NP-complete for every family $\mathcal{F}$ that includes only graphs of minimum vertex degree two (see Theorem 4.1). We mention in passing that, from this peculiar difference of the complexity of Vertex Cover and Secluded Vertex Cover, it would be interesting to find properties of $\mathcal{F}$ which govern whether Secluded $\mathcal{F}$-free Vertex Deletion is NP-hard or polynomial-time solvable along the lines of the well-known dichotomy results [17, 8].

▶ **Theorem 4.1** ($\star$). *For each family $\mathcal{F}$ containing only graphs of minimum vertex degree two,* Secluded $\mathcal{F}$-Free Vertex Deletion *is NP-complete.*

It is easy to see that Secluded $\mathcal{F}$-free Vertex Deletion is fixed-parameter tractable, more specifically, solvable in $c^k \cdot \mathrm{poly}(n)$ time: simply enumerate all inclusion-minimal $\mathcal{F}$-free vertex deletion sets S of size at most k using the standard search tree algorithm described by Cai [4] and check $|N[S]| \leq k$ for each of them. This works because, for any $\mathcal{F}$-free vertex deletion set S with $|N[S]| \leq k$, we can assume that S is an inclusion-minimal $\mathcal{F}$-free vertex deletion set since $|N[S']| \leq |N[S]|$ for any $S' \subsetneq S$.

We complement this observation of fixed-parameter tractability by the following kernelization result.

▶ **Theorem 4.2.** Secluded $\mathcal{F}$-free Vertex Deletion *has a problem kernel comprising $O(k^{c+1})$ vertices, where c is the maximum number of vertices in any graph of $\mathcal{F}$.*

Our proof of Theorem 4.2 will exploit *expressive* kernelization algorithms for d-Hitting Set [2, 3, 7], which maintain inclusion-minimal solutions and that return subgraphs of the input hypergraph as kernels: Herein, given hypergraph $H = (U, \mathcal{C})$ with $|C| \leq d$ for each $C \in \mathcal{C}$, and an integer k, d-Hitting Set asks whether there is a *hitting set* $S \subseteq U$ with $|S| \leq k$, that is, $C \cap S \neq \emptyset$ for each $C \in \mathcal{C}$. Our kernelization for Secluded $\mathcal{F}$-free Vertex Deletion is based on transforming the input instance (G, k) to a d-Hitting Set instance (H, k), computing an expressive d-Hitting Set problem kernel (H', k), and outputting a Secluded $\mathcal{F}$-free Vertex Deletion instance (G', k), where G' is the graph induced by the vertices remaining in H' together with at most $k + 1$ additional neighbors of each vertex in G.

▶ **Definition 4.3.** Let $(G = (V, E), k)$ be an instance of Secluded $\mathcal{F}$-free Vertex Deletion. For a vertex $v \in V$, let $N_j(v) \subseteq N_G(v)$ be j arbitrary neighbors of v, or $N_j(v) := N_G(v)$ if v has degree less than j. For a subset $S \subseteq V$, let $N_j(S) := \bigcup_{v \in S} N_j(v)$. Moreover, let

$c :=$ $\max_{F \in \mathcal{F}} |V(F)|$ be the maximum number of vertices in any graph in $\mathcal{F}$,
$H =$ $(U, \mathcal{C})$ be the hypergraph with $U := V$ and $\mathcal{C} := \{S \subseteq V \mid G[S] \in \mathcal{F}\}$,
$H' =$ $(U', \mathcal{C}')$ be a subgraph of H with $|U'| \in O(k^c)$ such that any set S with $|S| \leq k$ is an inclusion-minimal hitting set for H if and only if it is for H', and
$G' =$ (V', E') be the subgraph of G induced by $U' \cup N_{k+1}(U')$.

To prove Theorem 4.2, we show that (G', k) is a problem kernel for the input instance (G, k). The subgraph H' exists and is computable in linear time from H [3, 7]. Moreover, for constant c, one can compute H from G and G' from H' in polynomial time. It is obvious that the number of vertices of G' is $O(k^{c+1})$. Hence, it remains to show that (G', k) is a yes-instance if and only if (G, k) is. This is achieved by the following two lemmas.

▶ **Lemma 4.4.** *For any $S \subseteq U'$ with $|N_{G'}[S]| \leq k$, it holds that $N_G[S] = N_{G'}[S]$.*

Proof. Since $S \subseteq U' \subseteq V' \cap V$ and since G' is a subgraph of G, it is clear that $N_G[S] \supseteq N_{G'}[S]$. For the opposite direction, observe that each $v \in S$ has degree at most k in G'. Thus, v has degree at most k in G since, otherwise, $k + 1$ of its neighbors would be in G' by construction. Thus, $N_{G'}(v) \supseteq N_{k+1}(v) = N_G(v)$ for all $v \in S$ and, thus, $N_{G'}[S] \supseteq N_G[S]$. ◀

▶ **Lemma 4.5** ($\star$). *Graph G allows for an $\mathcal{F}$-free vertex deletion set S with $|N_G[S]| \leq k$ if and only if G' allows for an $\mathcal{F}$-free vertex deletion set S with $|N_{G'}[S]| \leq k$.*

4.2 Small Secluded $\mathcal{F}$-free Vertex Deletion

In this subsection, we present a fixed-parameter algorithm for the following problem parameterized by $\ell + k$.

Small Secluded $\mathcal{F}$-free Vertex Deletion
Input: A graph $G = (V, E)$, and two integers k, ℓ.
Question: Is there a subset $S \subseteq V$ such that $G - S$ is $\mathcal{F}$-free, $|S| \leq k$, and $|N_G(S)| \leq \ell$?

As before, we call a set $S \subseteq V$ such that $G - S$ is $\mathcal{F}$-free an *$\mathcal{F}$-free vertex deletion set.*

In the previous section, we discussed a simple search tree algorithm for Secluded $\mathcal{F}$-free Vertex Deletion that was based on the fact that we could assume that our solution is an inclusion-minimal $\mathcal{F}$-free vertex deletion set. However, an $\mathcal{F}$-free vertex deletion set S with

$|S| \leq k$ and $|N_G(S)| \leq \ell$ is not necessarily inclusion-minimal: some vertices may have been added to S just in order to shrink its open neighborhood. However, the following simple lemma limits the number of possible candidate vertices that can be used to enlarge S in order to shrink $N(S)$, which we will use in a branching algorithm.

▶ **Lemma 4.6.** *Let S be an $\mathcal{F}$-free vertex deletion set and $S' \supseteq S$ such that $|S'| \leq k$ and $|N_G(S')| \leq \ell$, then $|N_G(S)| \leq \ell + k$.*

Proof. $|N_G(S)| = |N_G[S] \setminus S| \leq |N_G[S'] \setminus S| \leq |N_G[S']| \leq |N_G(S') \cup S'| \leq \ell + k$. ◀

▶ **Theorem 4.7.** Small Secluded $\mathcal{F}$-free Vertex Deletion *can be solved in* $\max\{c, k + \ell\}^k \cdot \mathrm{poly}(n)$*-time, where c is the maximum number of vertices in any graph of $\mathcal{F}$.*

Proof. First enumerate all inclusion-minimal $\mathcal{F}$-free vertex deletion sets S with $|S| \leq k$. This is possible in $c^k \cdot \mathrm{poly}(n)$ time using the generic search tree algorithm described by Cai [4]. For each $k' \leq k$, this search tree algorithm generates at most $c^{k'}$ sets of size k'. Now, for each enumerated set S of k' elements, do the following:

1. If $|N_G(S)| \leq \ell$ then output S as our solution.
2. If $|N_G(S)| > \ell + k$, then S cannot be part of a solution S' with $N_G(S') \leq \ell$ by Lemma 4.6, we proceed with the next set.
3. Otherwise, initiate a recursive branching: recursively branch into at most $\ell + k$ possibilities of adding a vertex from $N_G(S)$ to S as long as $|S| \leq k$.

The recursive branching initiated at step 3 stops at depth $k - k'$ since, after adding $k - k'$ vertices to S, one obtains a set of size k. Hence, the total running time of our algorithm is

$$\mathrm{poly}(n) \cdot \sum_{k'=1}^{k} c^{k'} (\ell + k)^{k-k'} = \mathrm{poly}(n) \cdot \sum_{k'=1}^{k} \max\{c, \ell + k\}^k = \mathrm{poly}(n) \cdot \max\{c, \ell + k\}^k. \quad ◀$$

Given Theorem 4.7, a natural question is whether the problem allows for a polynomial kernel.

5 Feedback Vertex Set

In this section we consider the Feedback Vertex Set problem, that asks, given a graph G and an integer k, whether there exists $W \subseteq V(G)$, $|W| \leq k$, such that $G - W$ is cycle-free.

5.1 Secluded Feedback Vertex Set

We show in this subsection that the problem below is NP-hard and admits a polynomial kernel.

Secluded Feedback Vertex Set (SFVS)
Input: A graph $G = (V, E)$, and an integer k.
Question: Is there a set $S \subseteq V$ such that $G - S$ is cycle-free and $|N_G[S]| \leq k$?

▶ **Theorem 5.1** ($\star$). Secluded Feedback Vertex Set *is* NP*-hard.*

▶ **Theorem 5.2** ($\star$). Secluded Feedback Vertex Set *admits a kernel with $O(k^5)$ vertices.*

5.2 Small Secluded Feedback Vertex Set

In this section, we show that the following problem is unlikely to be fixed-parameter tractable with respect to the parameter ℓ.

SMALL SECLUDED FEEDBACK VERTEX SET
Input: A graph $G = (V, E)$, and two integers k, ℓ.
Question: Is there a set $S \subseteq V$ such that $G - S$ is cycle-free, $|S| \leq k$, and $|N_G(S)| \leq \ell$?

▶ **Theorem 5.3** (⋆). SMALL SECLUDED FEEDBACK VERTEX SET *is W[1]-hard with respect to* ℓ.

6 Independent Set

For INDEPENDENT SET, being a maximization problem, it makes little sense to bound the size of the closed neighborhood from above, as in this case the empty set always constitutes a solution. One might ask for an independent set with closed neighborhood as large as possible. However, the closed neighbor of any inclusion-wise maximal independent set comprise the whole vertex set (as otherwise the independent set would not be maximal). Hence, this question is also trivial.

Therefore, in this section we only consider the following problem.

LARGE SECLUDED INDEPENDENT SET (LSIS)
Input: A graph $G = (V, E)$ and two integers k, ℓ.
Question: Is there an independent set $S \subseteq V$ such that $|S| \geq k$ and $|N_G(S)| \leq \ell$?

A simple reduction from standard INDEPENDENT SET shows that LSIS is $W[1]$-hard with respect to k. We show that this is the case also when parameterized by $k + \ell$.

▶ **Theorem 6.1** (⋆). LARGE SECLUDED INDEPENDENT SET *is* $W[1]$*-hard with respect to* $k+\ell$.

7 Summary and Future Work

In this paper we studied the problem of finding sets of vertices in a graph that fulfill certain properties and have a small neighborhood. We present computational complexity results for secluded and small secluded variants of s-t SEPARATOR, q-DOMINATING SET, FEEDBACK VERTEX SET, $\mathcal{F}$-FREE VERTEX DELETION, and for the large secluded variant of INDEPENDENT SET. In the case of s-t SEPARATOR, we leave as an open question the parameterized complexity of SMALL SECLUDED s-t SEPARATOR with respect to $k + \ell$ or k. We conjecture that it is fixed-parameter tractable when parameterized by $k + \ell$. Concerning SECLUDED $\mathcal{F}$-FREE VERTEX DELETION, we would like to point out that investigating more precisely which graph families $\mathcal{F}$ yield NP-hardness as opposed to polynomial-time solvability is an interesting area for future research.

A natural way to generalize our results would be to consider vertex-weighted graphs and directed graphs. This generalization was already investigated by Chechik et al. [5] for SECLUDED PATH and SECLUDED STEINER TREE. Furthermore, we would like to mention that replacing the bound on the open neighborhood in the case of small secludedness by a bound on the outgoing edges of a solution would be an interesting modification of the problem. The variation follows the idea of the concept of *isolation* as used e.g. in [15, 13, 16, 14]. As the number of outgoing edges is at least as large as the open neighborhood, this might offer new possibilities for fixed-parameter algorithms.

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