Temporal Role Colouring

Joint work with Jessica Enright, Kitty Meeks and Puck Rombach

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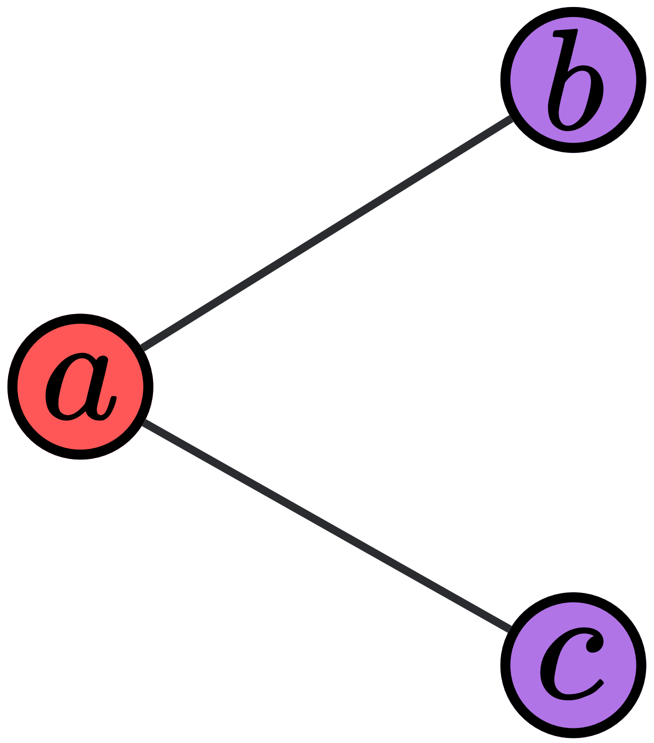
University of Glasgow

GRAPHS AND COLOURING

A graph is tuple $G = (V, E)$ where V is the set of *vertices* and E is the set of *edges*

$$V = \{a, b, c\}$$

$$E = \{ab, ac\}$$



A colouring α of G is a function assigning a label to each vertex of G

$$\alpha_i : V \to \{ \quad , \quad \}$$

$$\alpha_1(a) =$$

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$$\alpha_i : V \to \{\ {\color{red}\bullet}\ ,\ {\color{purple}\bullet}\ \}$$

$$\alpha_1(a) = {\color{red}\bullet} \qquad \alpha_2(a) = {\color{red}\bullet}$$

$$\alpha_1(b) = {\color{purple}\bullet} \qquad \alpha_2(b) = {\color{red}\bullet}$$

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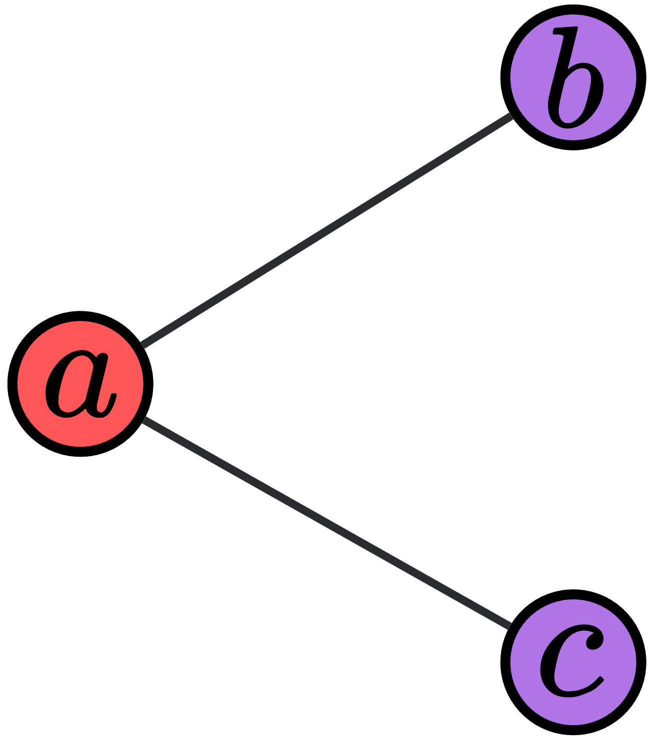
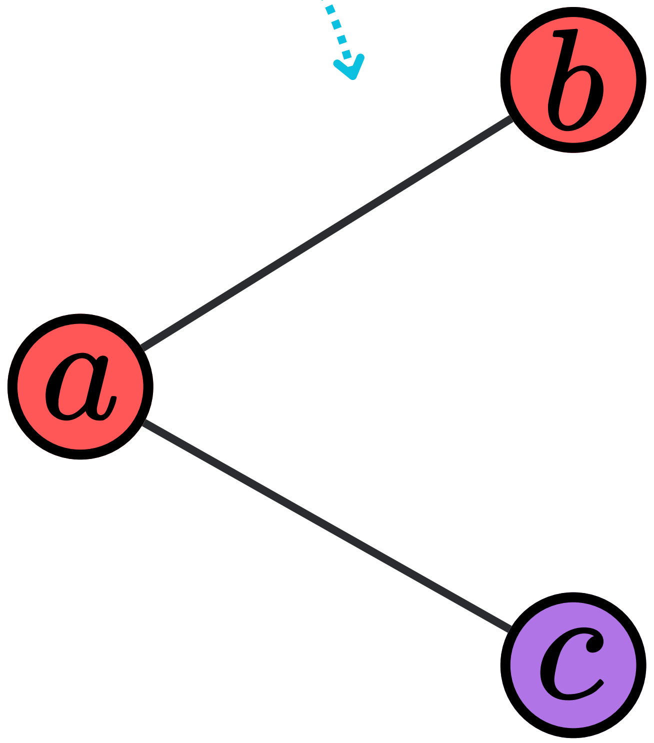
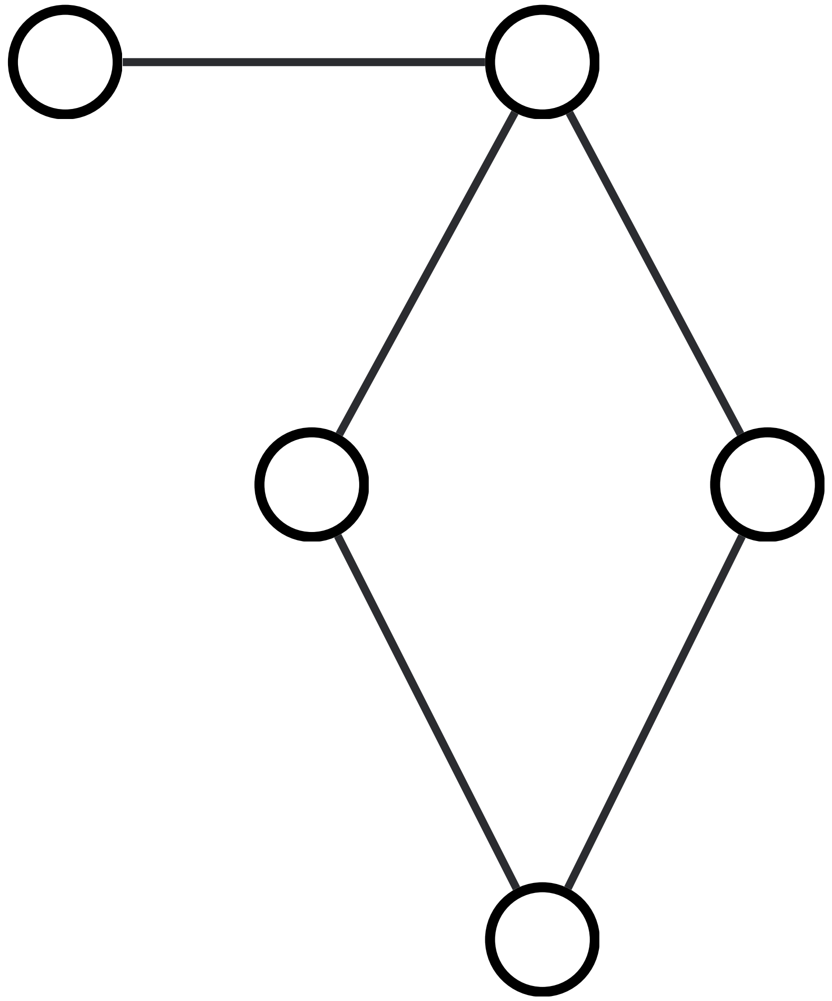


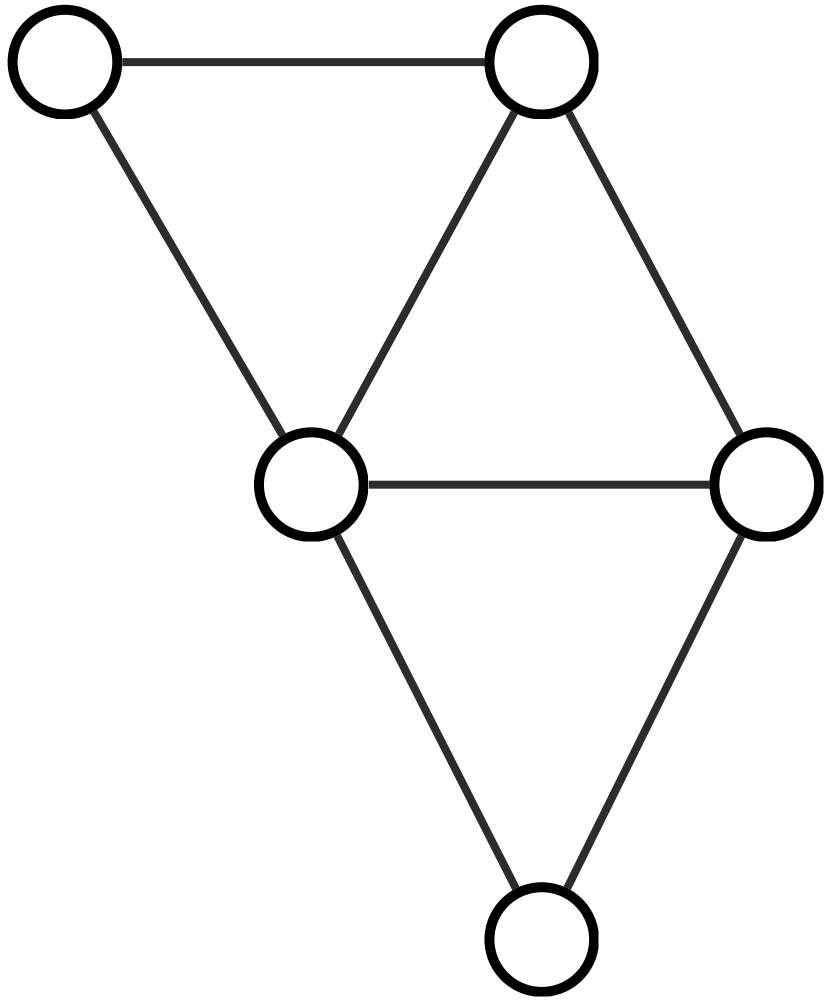


R-ROLE COLOURING

Given a graph $R = (V_R, E_R)$, an R-role colouring of G is a surjective map $\alpha : V_G \to V_R$ such that for all $v \in V_G, \alpha(N_G(v)) = N_R(\alpha(v))$



where $N_G(v)$ is the set of neighbours of v in G and $N_R(v)$ are the neighbours in R



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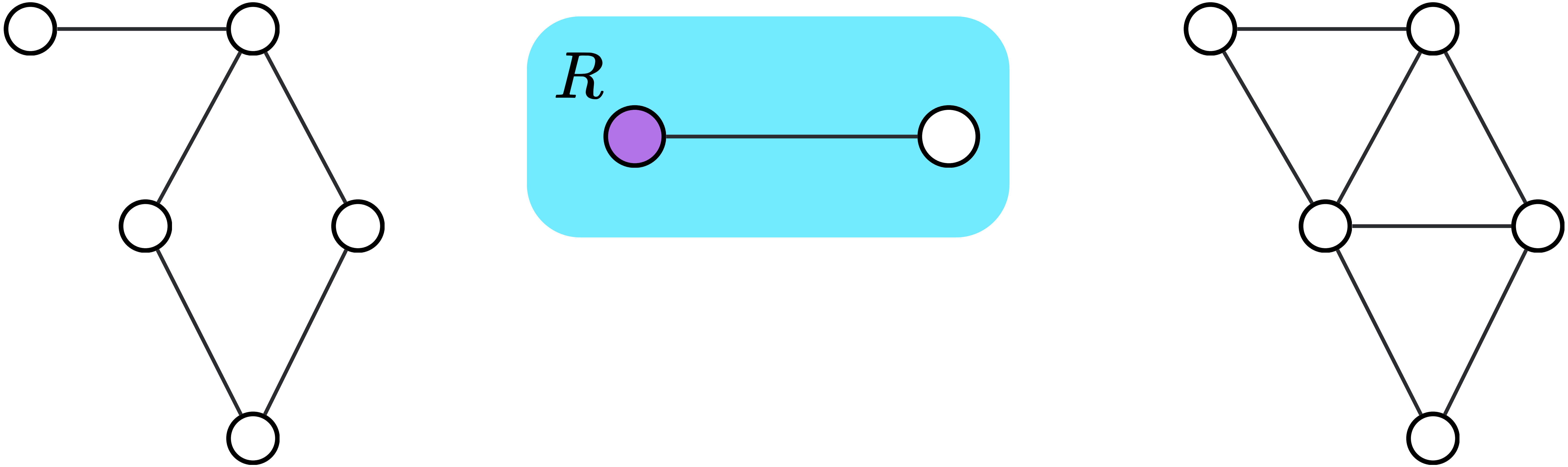

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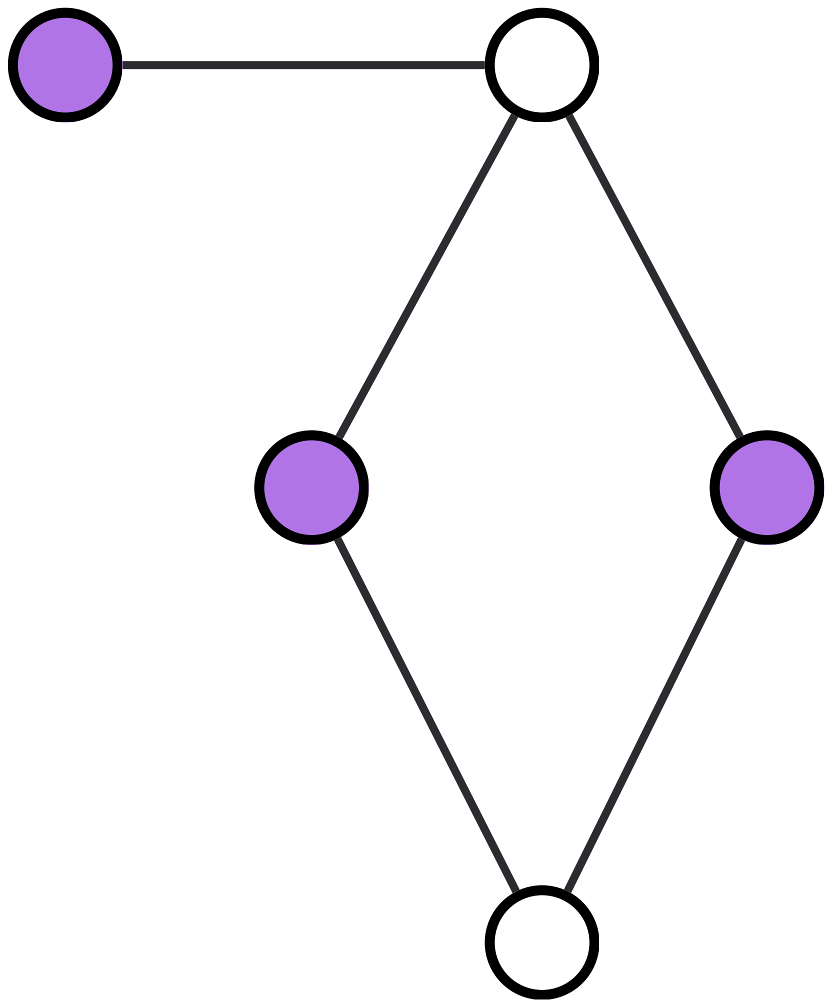

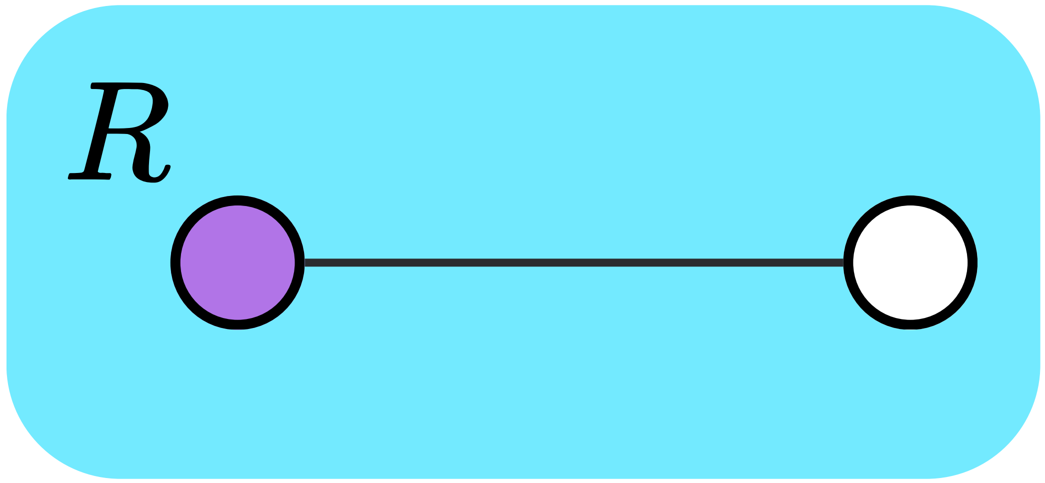

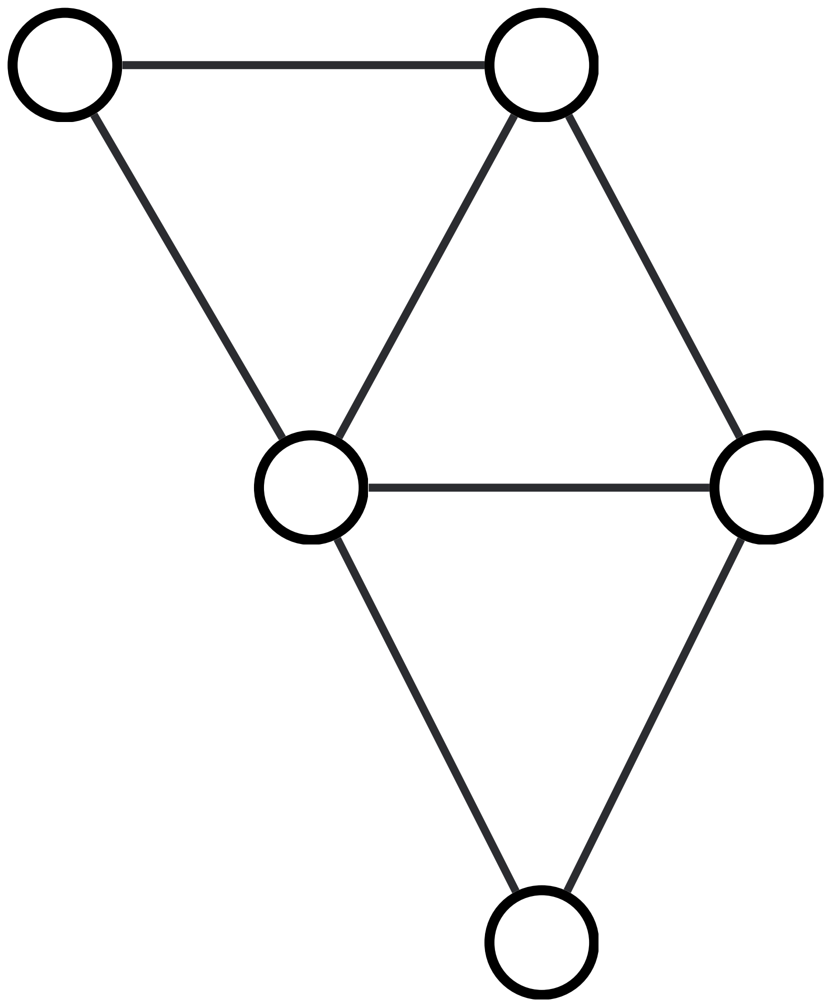

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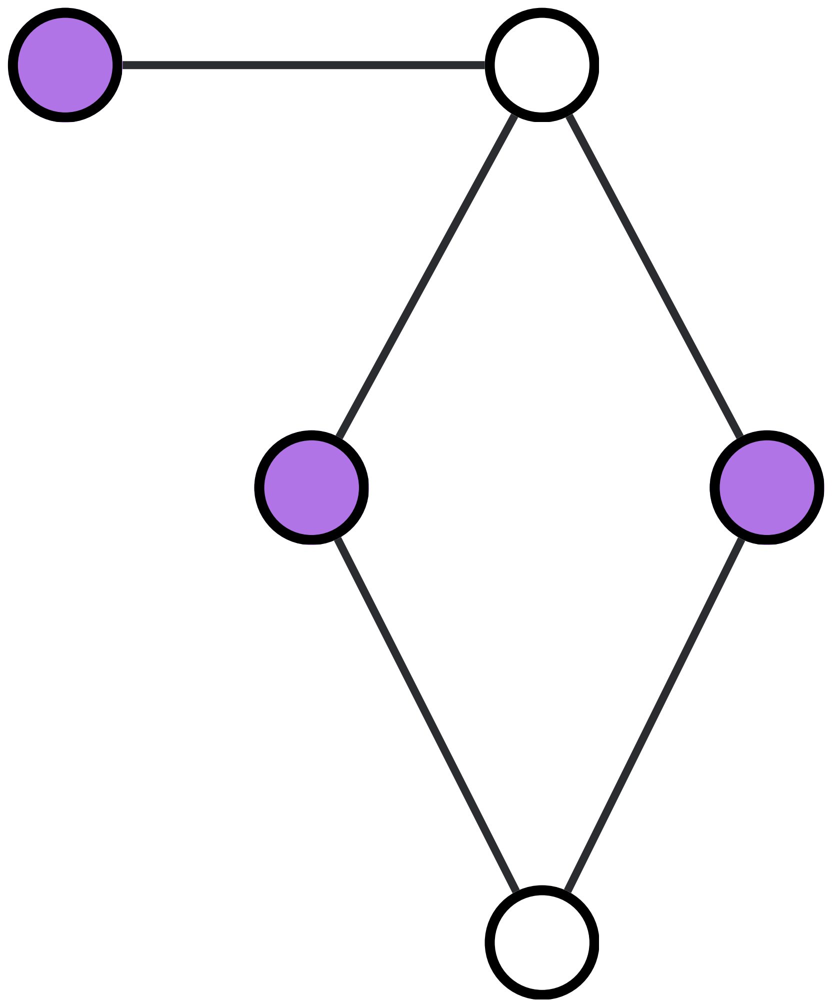

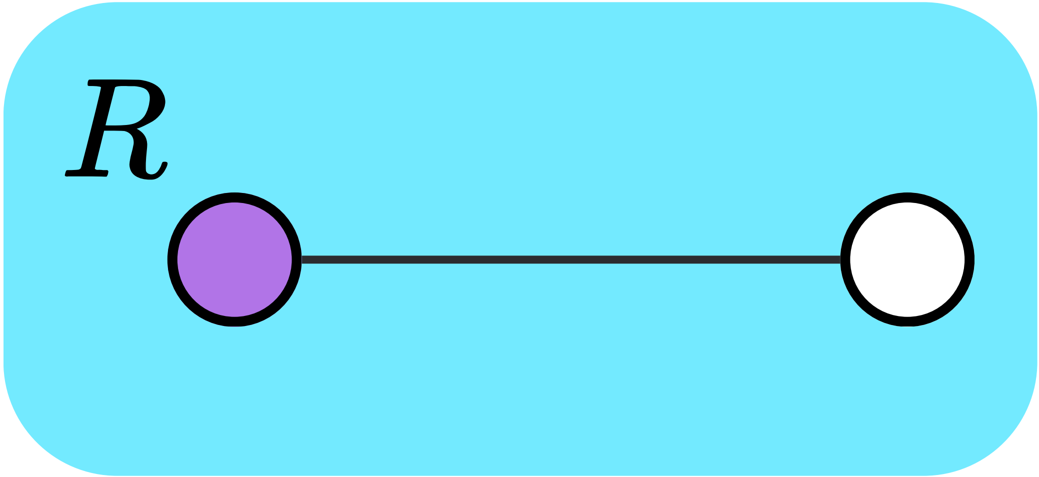

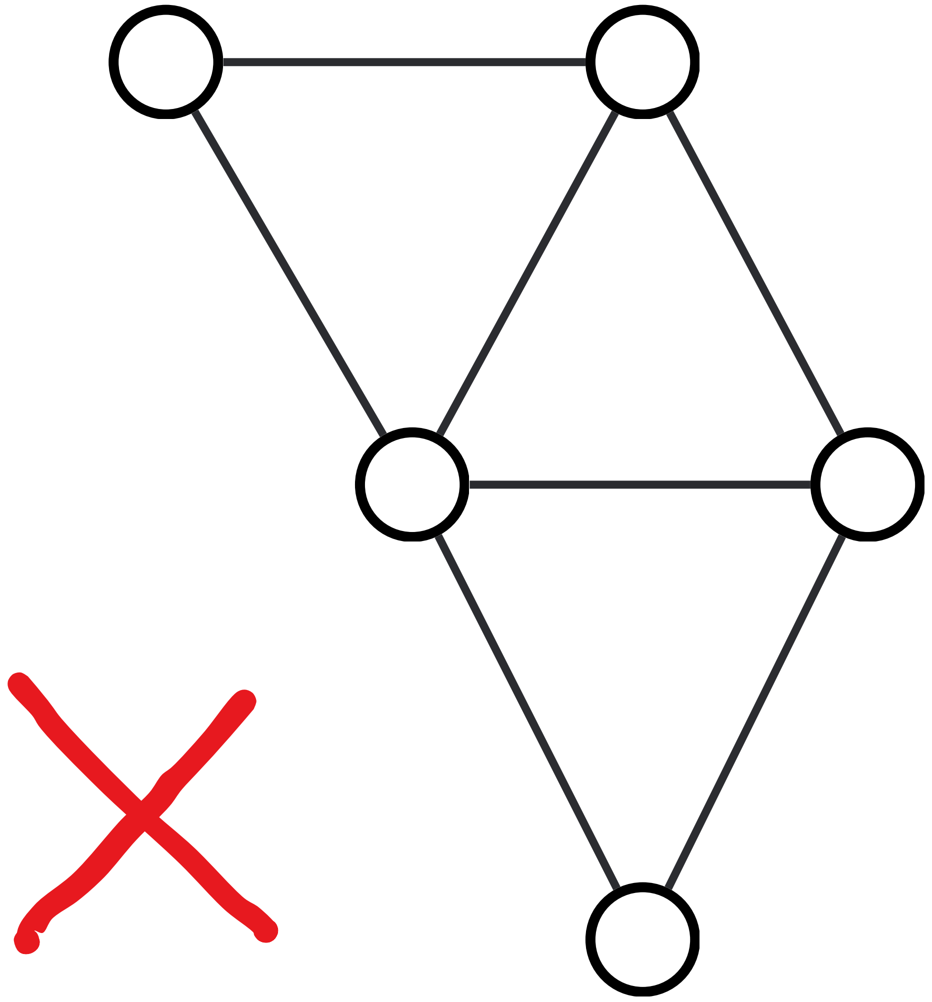

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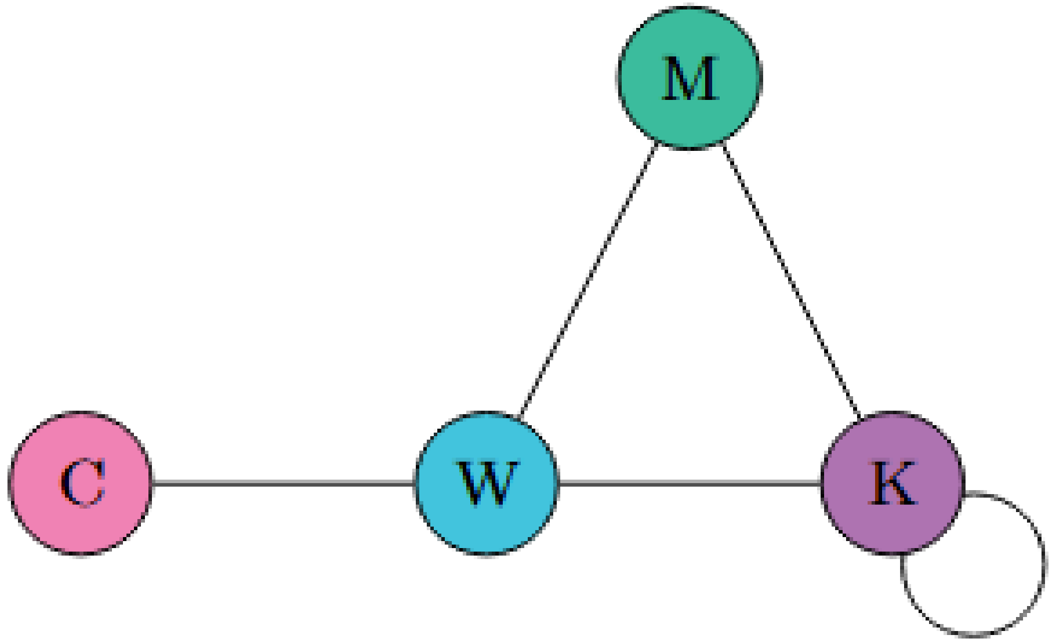
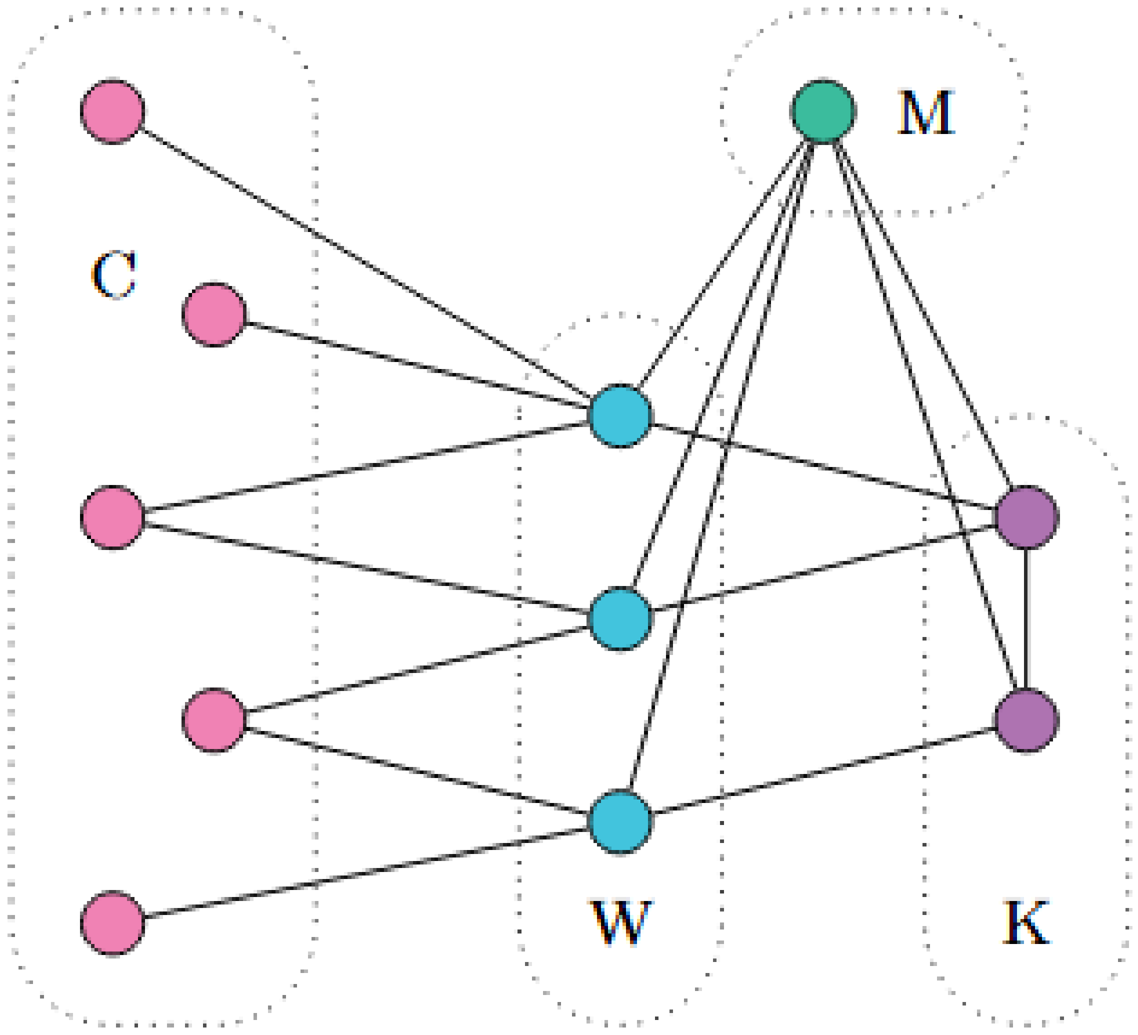
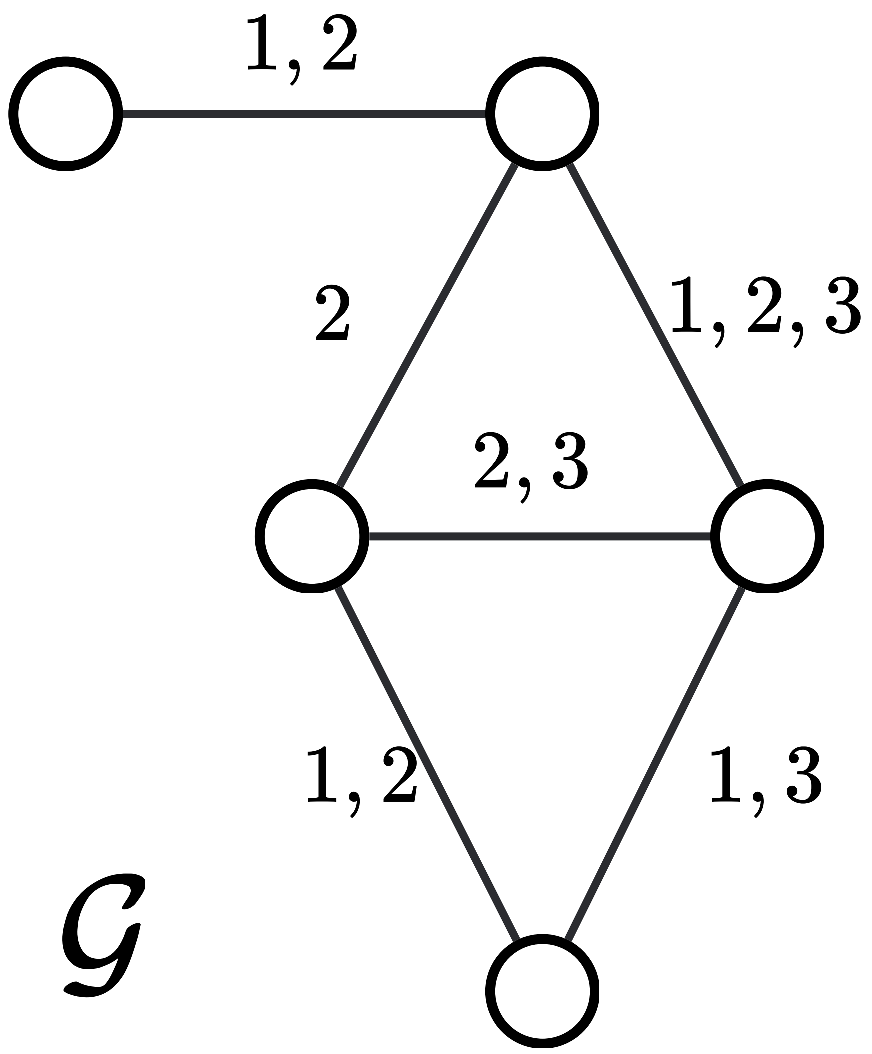




TEMPORAL GRAPHS

A *temporal graph* $\mathcal{G}$ is a tuple (G, λ) where $G = (V, E)$ is the *underlying graph* and $\lambda : E \to 2^{\Lambda}$ is a function assigning a set of times to each edge

Λ is the *lifetime* of the graph



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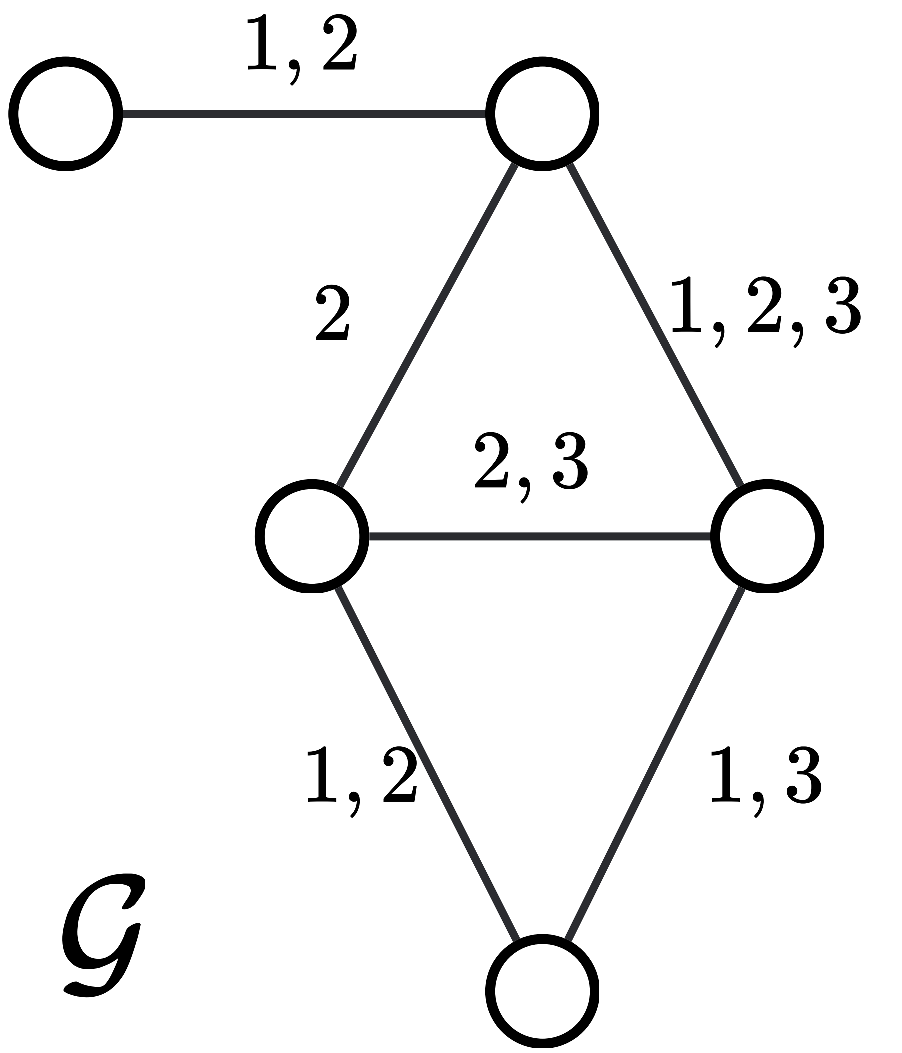

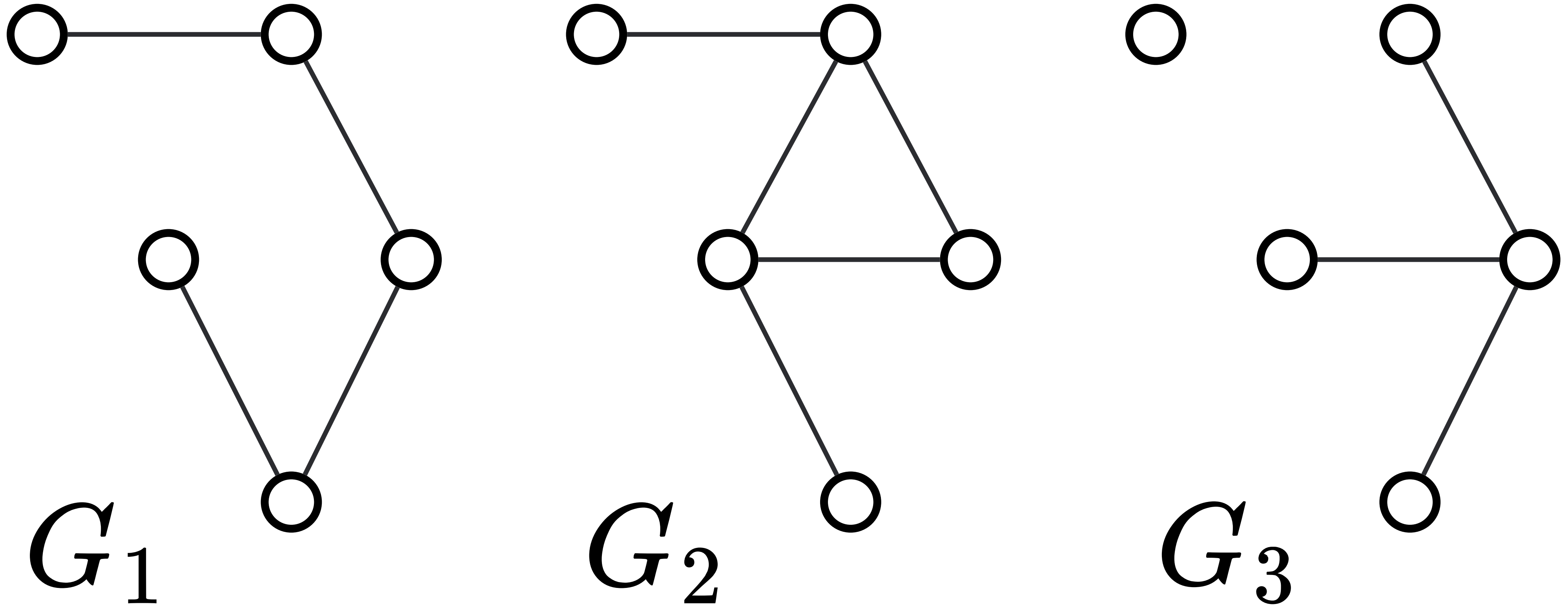

TEMPORAL ROLE COLOURING

Input: Temporal graph $\mathcal{G} = (G, \lambda)$, a k-colour role automaton A.

Question: Does $\mathcal{G}$ admit an A-role colouring?

TEMPORAL ROLE COLOURING

What does a *k-colour role automaton* look like?

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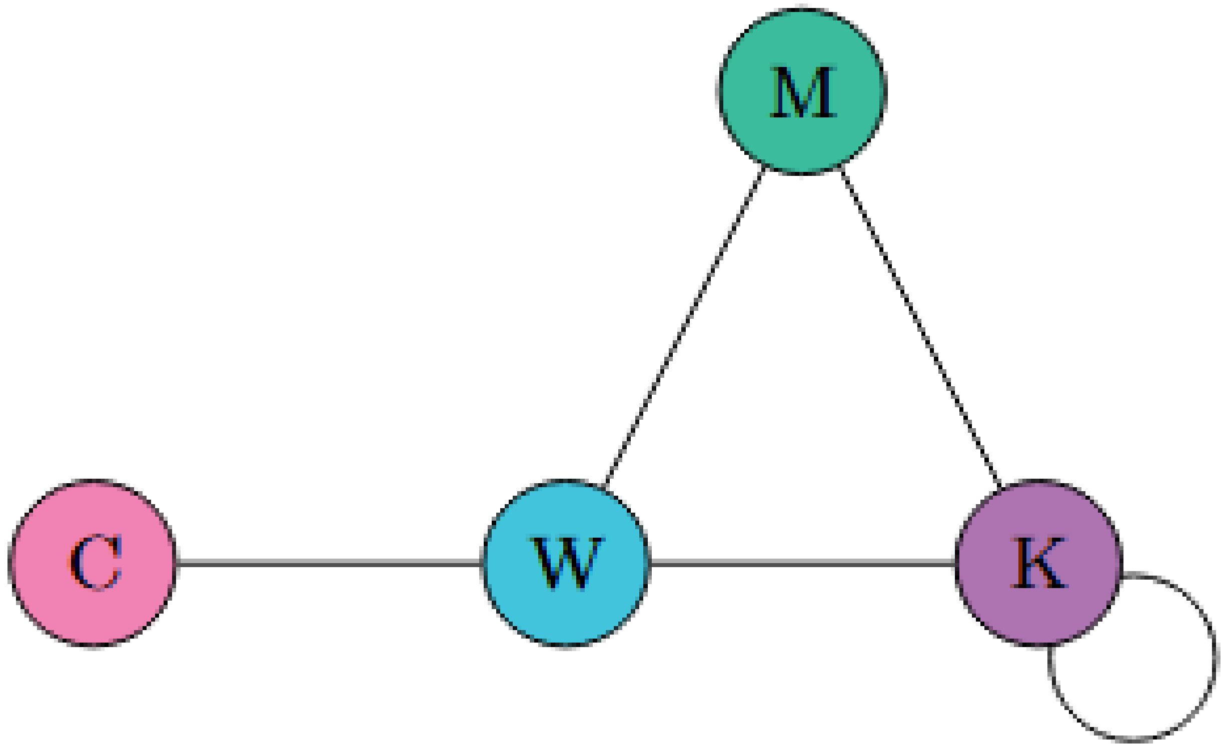

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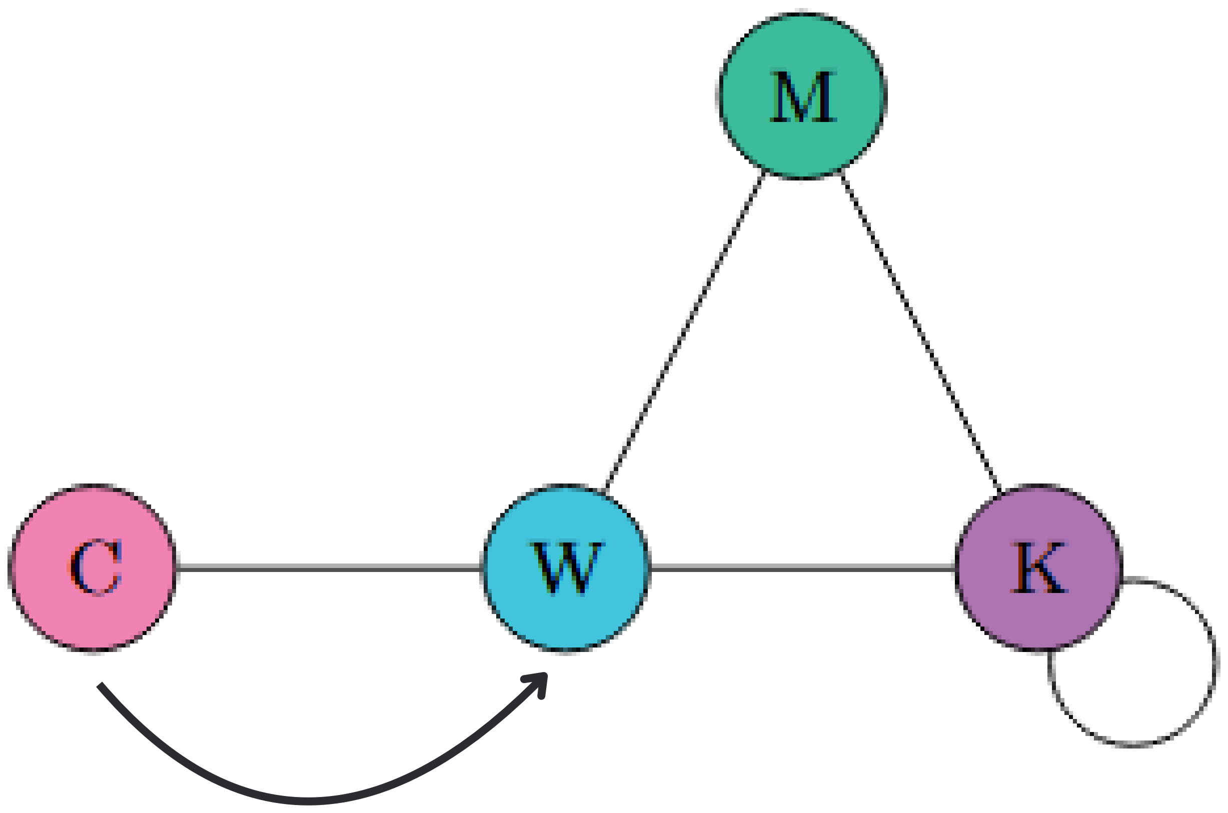

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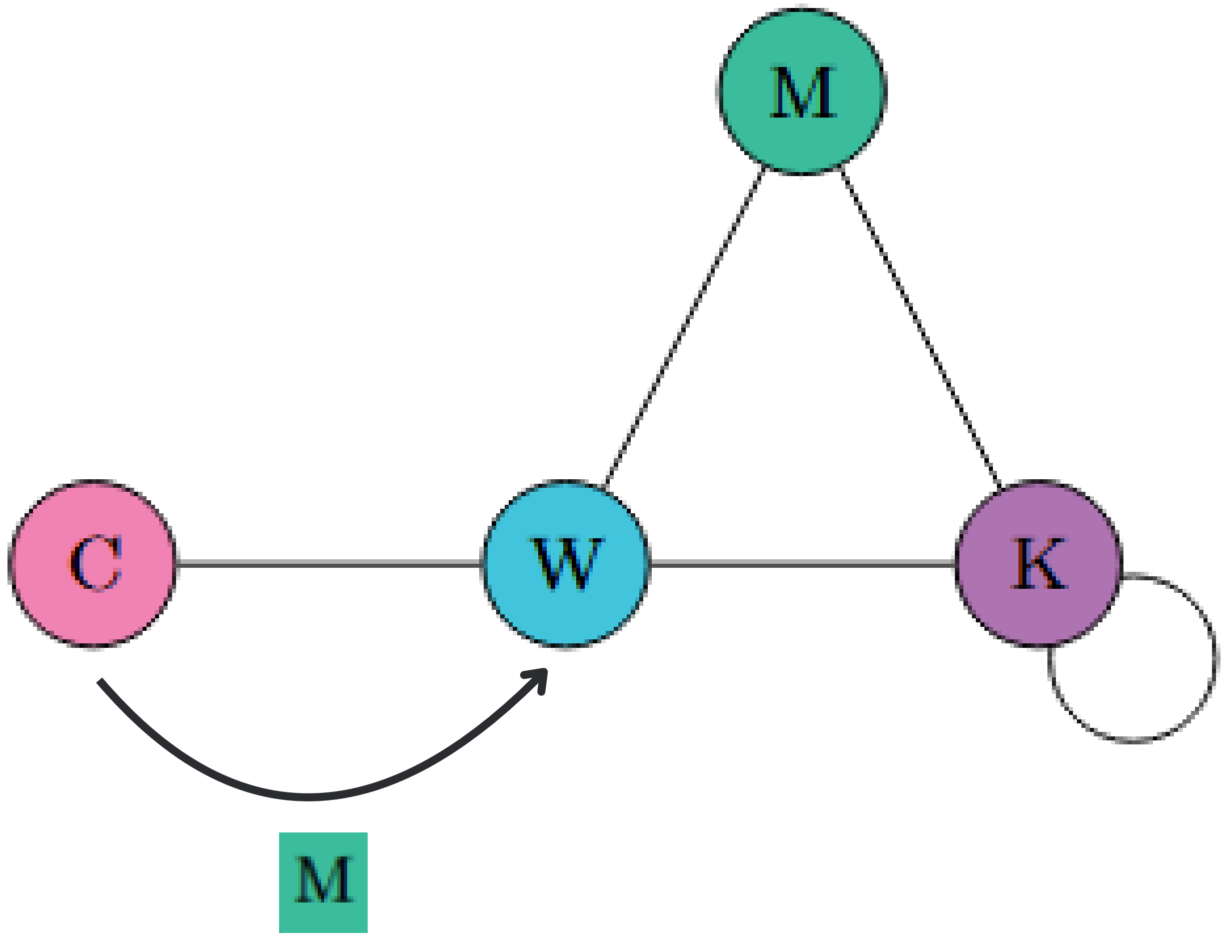


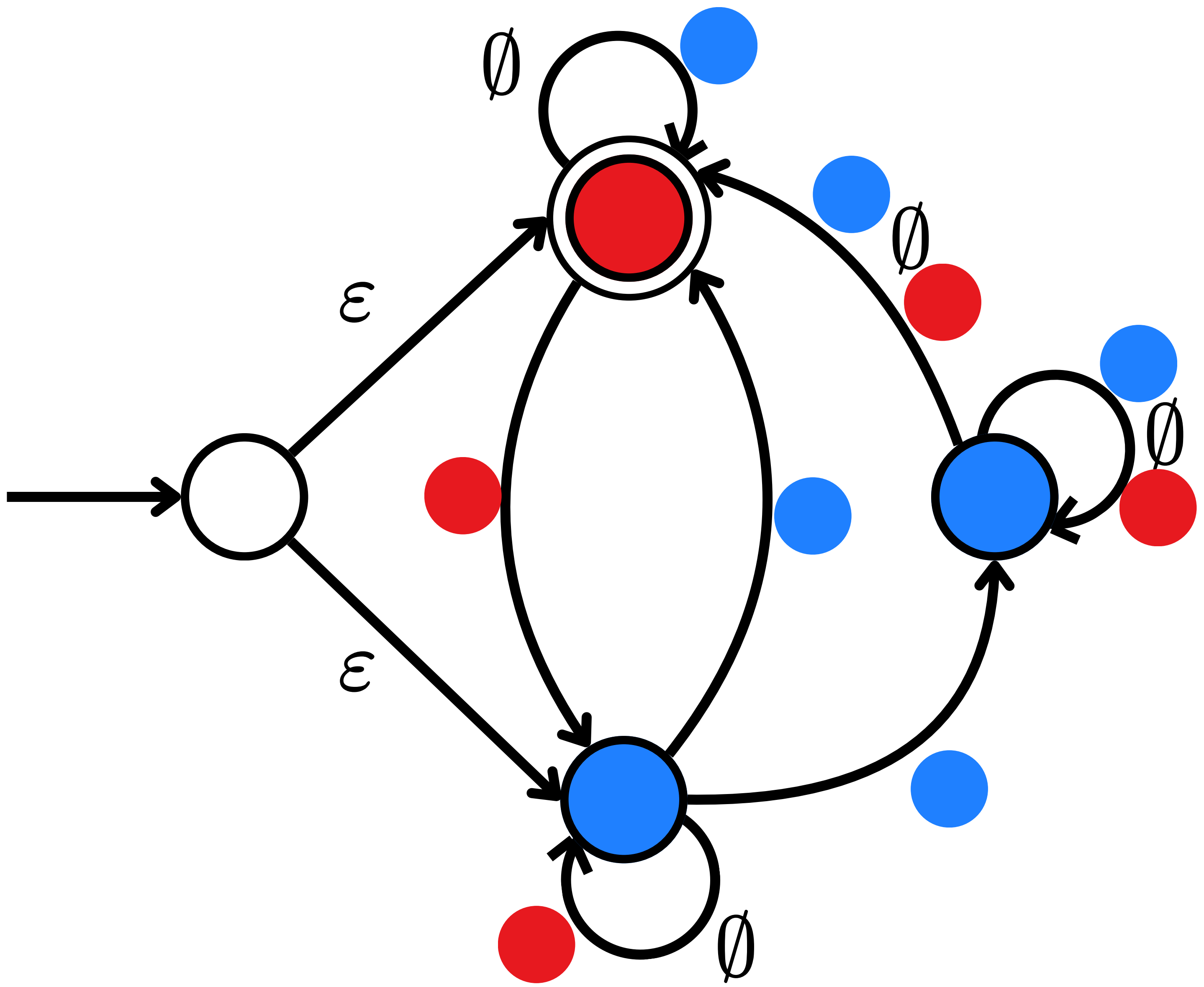
TEMPORAL ROLE COLOURING

For an integer k, A is a *k-colour role automaton* if it is a finite automaton with an additional colouring of states c. Specifically it is a tuple containing:

- An input alphabet $\Sigma = \mathcal{P}([k])$
- A set Q of states
- A colouring of states, $c : Q \to \{0, .., k\}$
- A set $F \subseteq Q$ of accepting states
- A start state q_0 such that $c(q_0) = 0$
- A transition function $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ where all transitions from q_0 are ε-transitions and there are no ε-transitions from any other state.

Example:

At most one colour of neighbours at a time
Start either colour
If red sees another red immediately turn blue
Once blue sees another blue it can turn red
All must be red at the end



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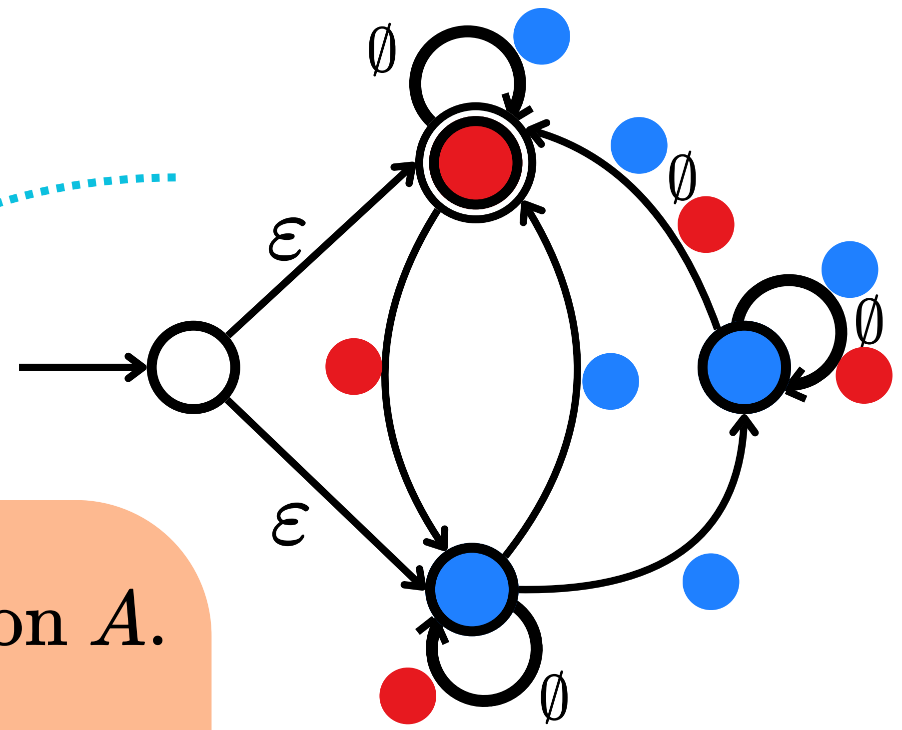

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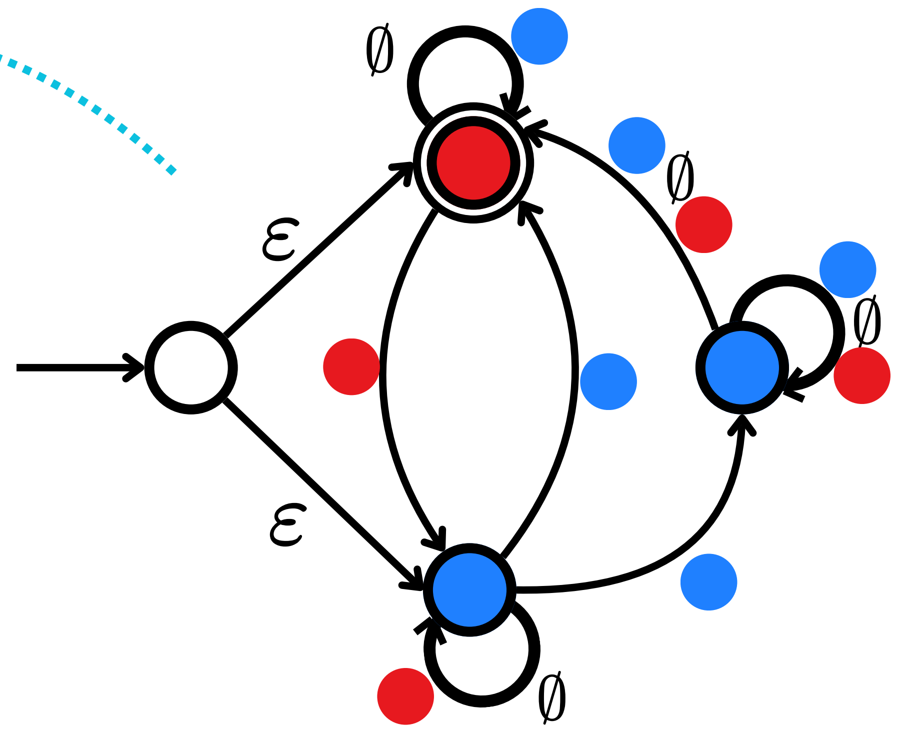

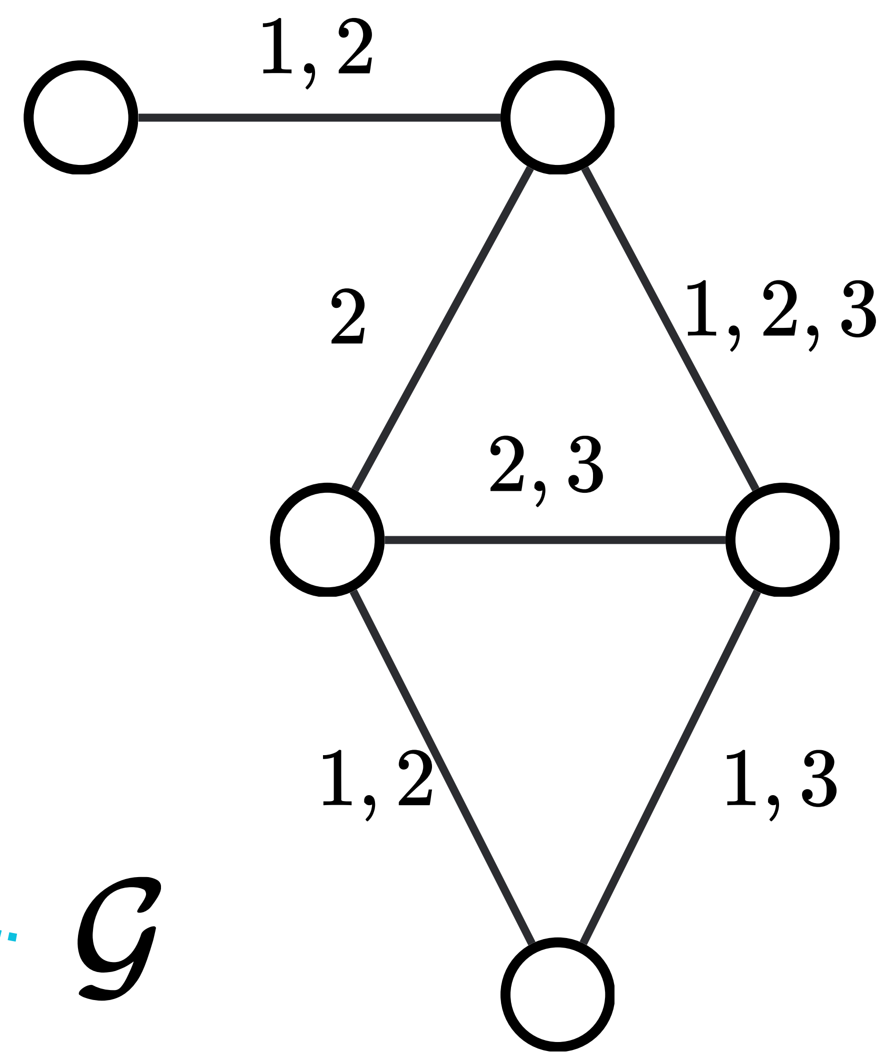

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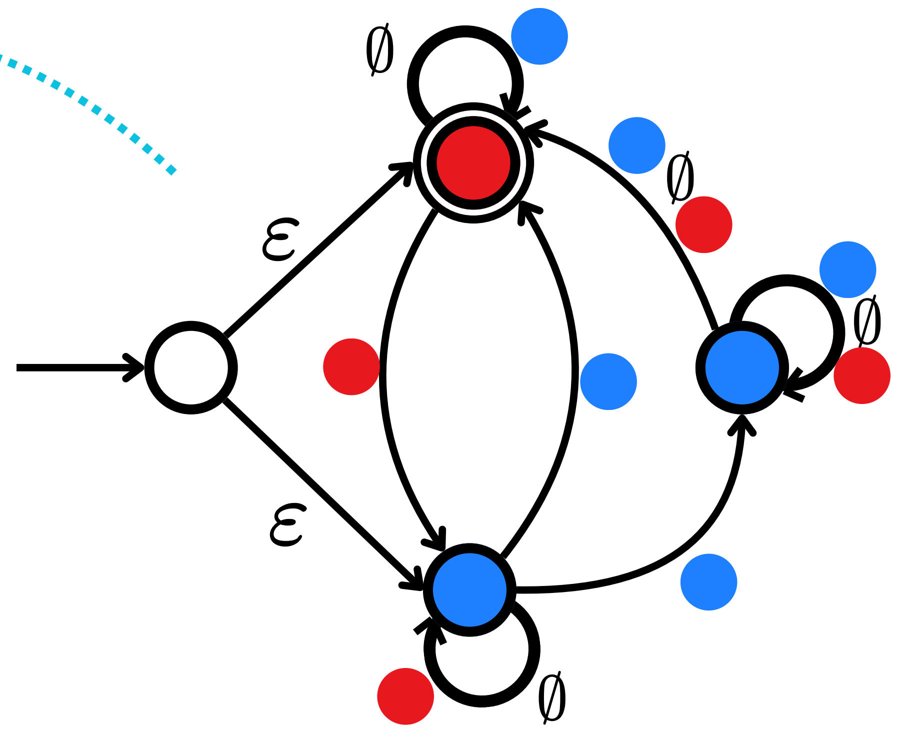

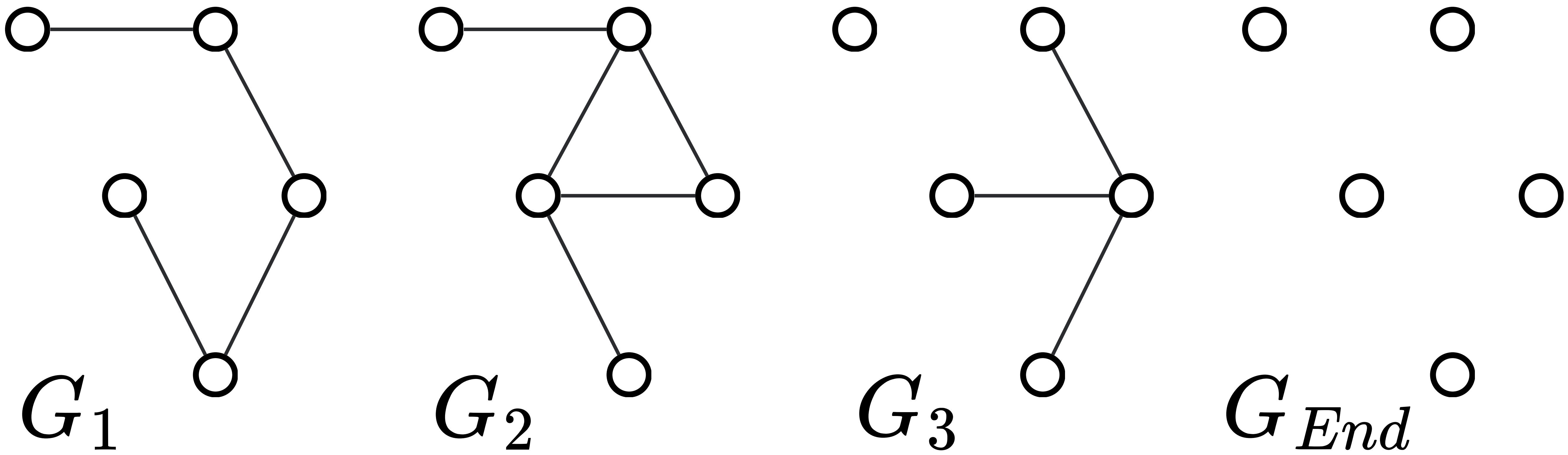

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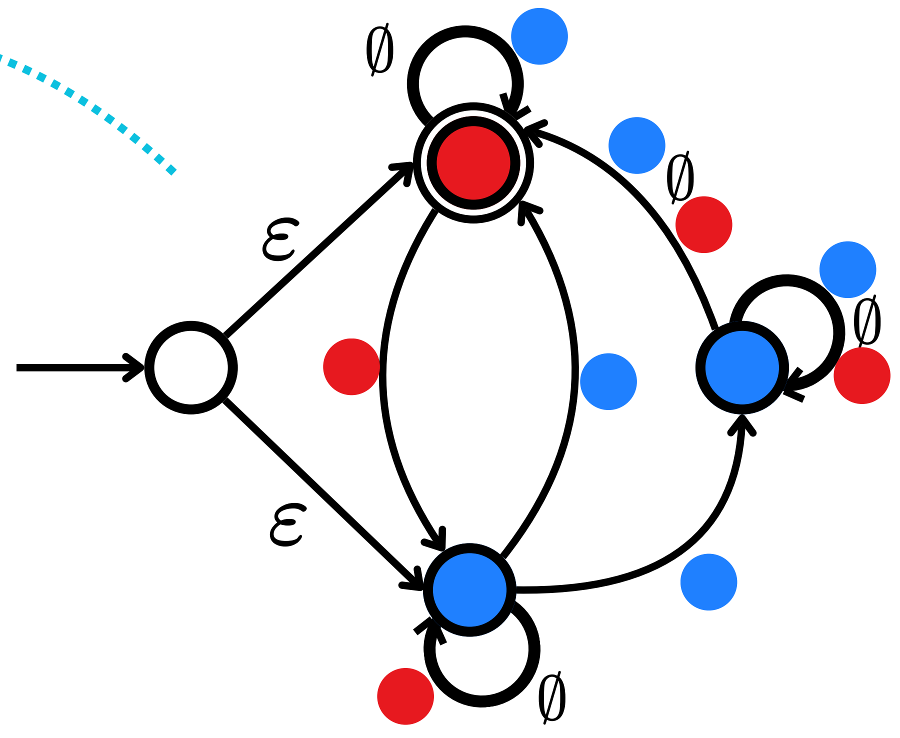

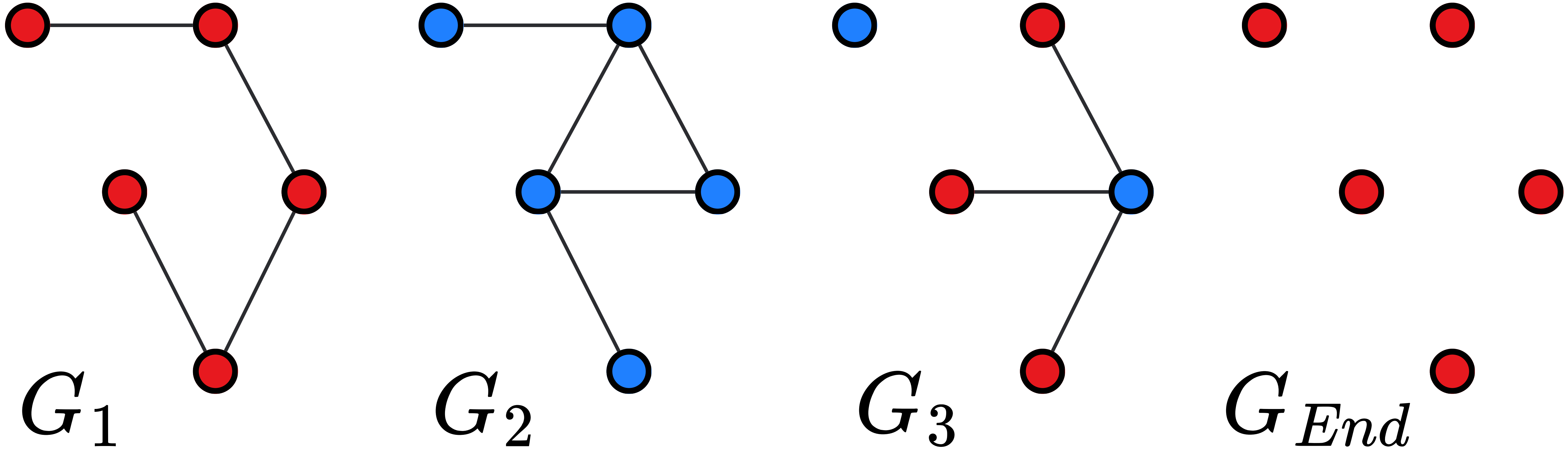

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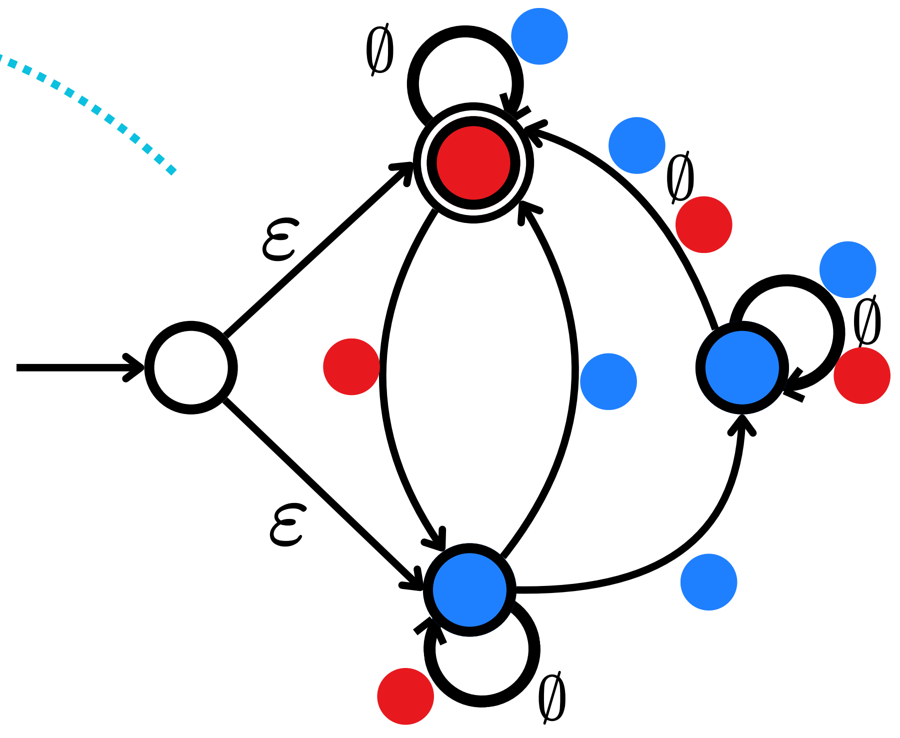

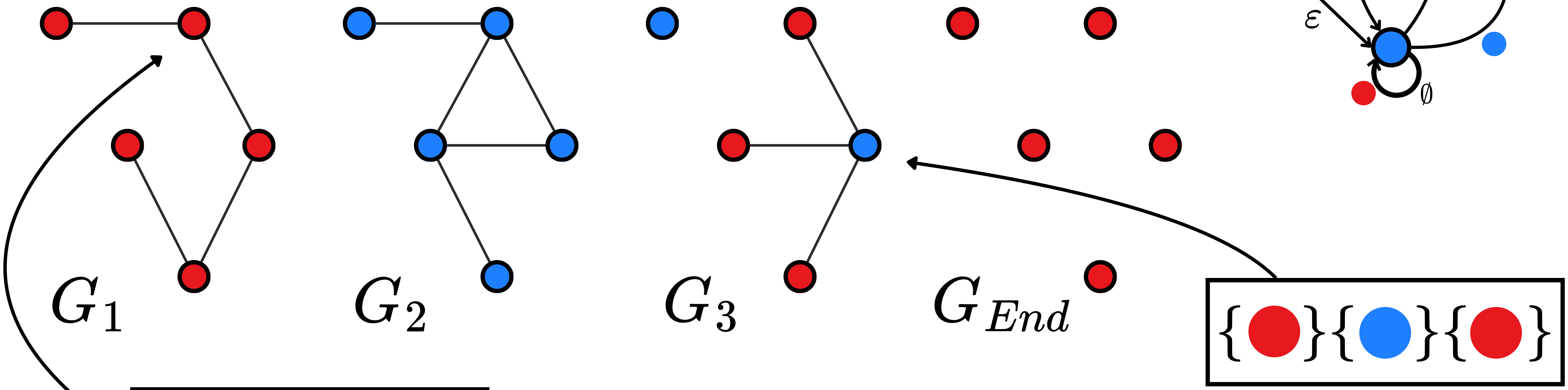

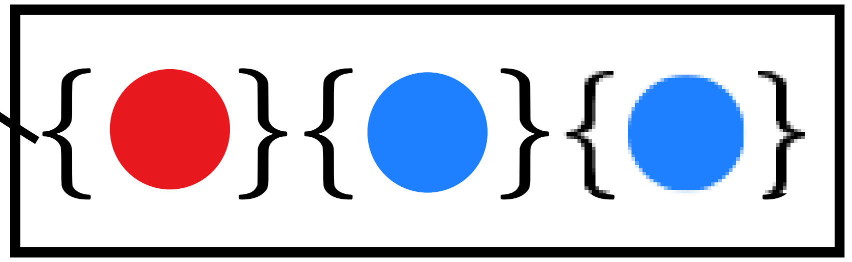

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NP-COMPLETENESS

- Input alphabet $\Sigma = 2^{[k]}$

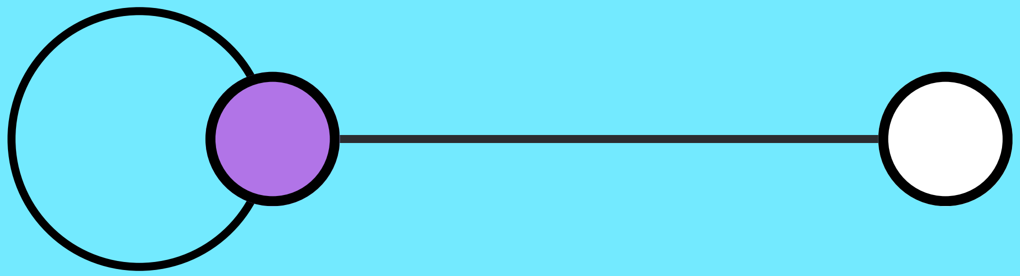
- A start state q_0

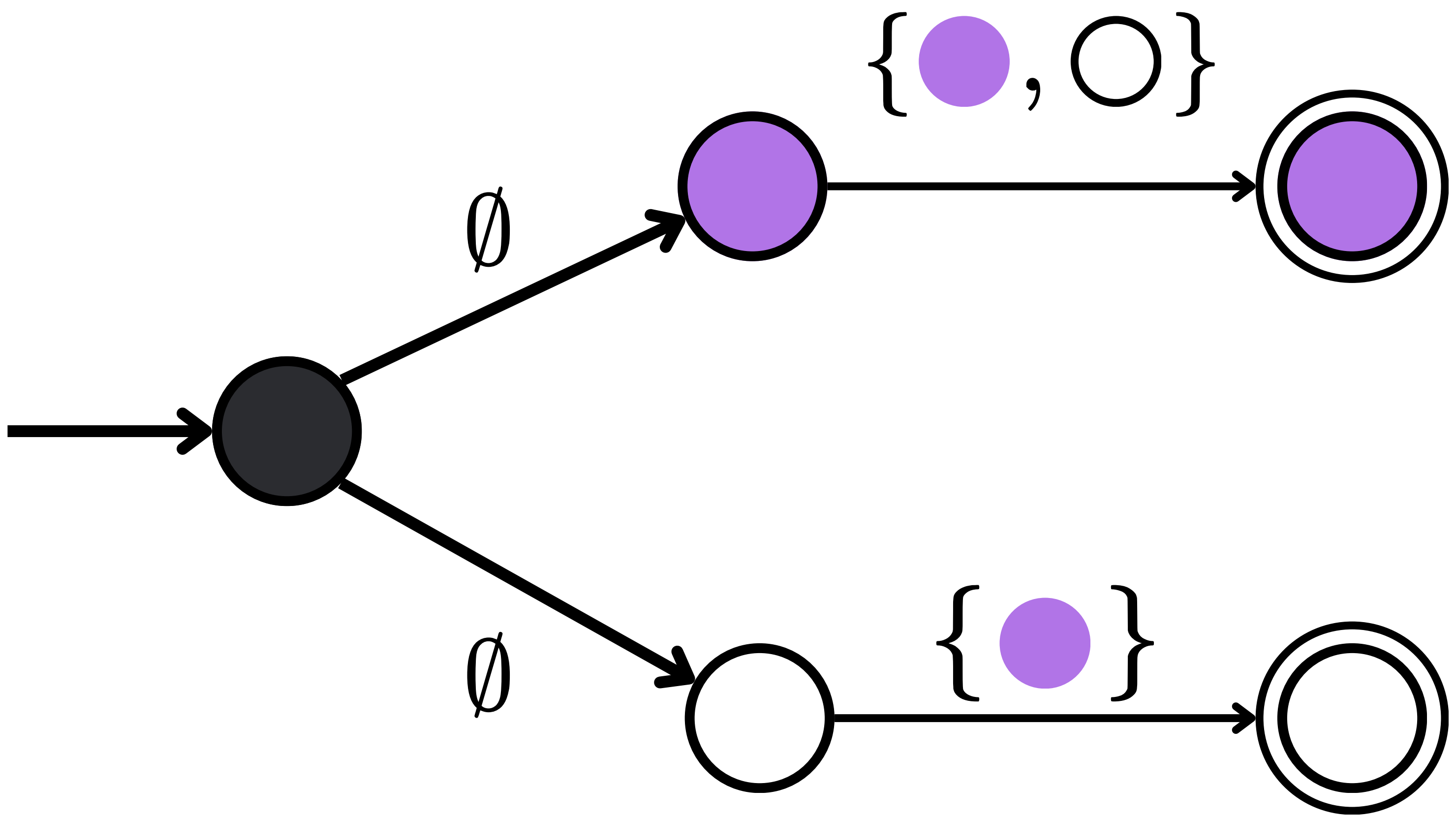
- A set of states $Q = \{q_0, q_{1,s}, q_{1,a}, q_{2,s}, q_{2,a}, \ldots, q_{k,s}, q_{k,a}\}$, that is, the start state and two states for each vertex in R

- A colouring of states $c : Q \to \{0, \ldots, k\}$ defined as $c(q_0) = 0$ and $c(q_{i,a}) = c(q_{i,s}) = i$

- A set $F = \{q_{1,a}, q_{2,a}, \ldots q_{k,a}\}$ of accepting states

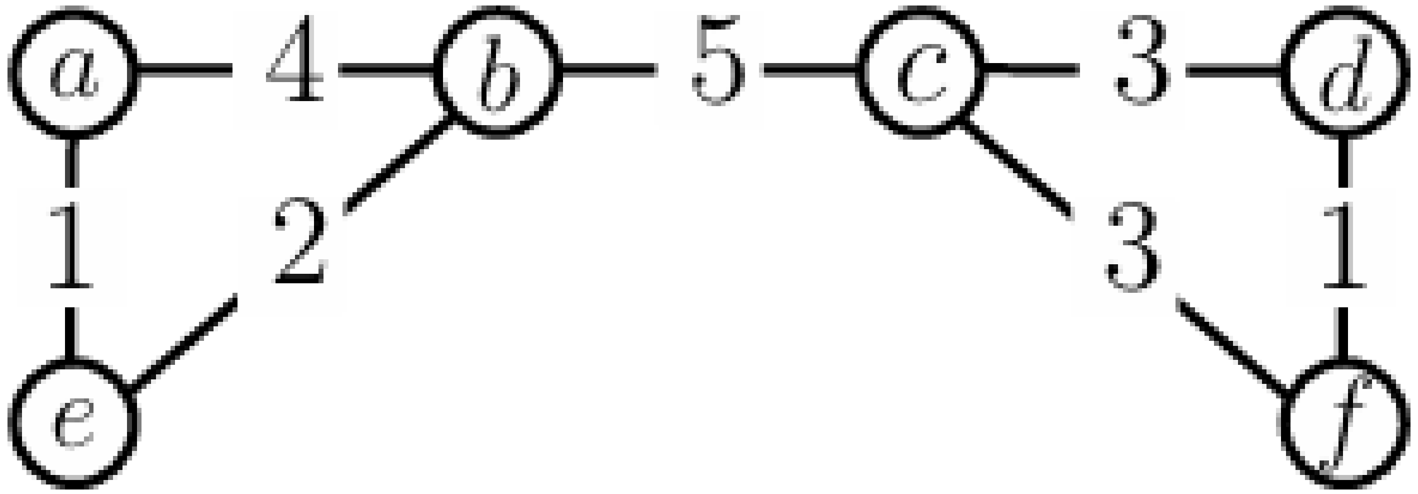
- A transition function δ with $\delta(q_0, \varepsilon) = q_{i,s}$ for each $i \in [k]$ and $\delta(q_{i,s}, N_R(i)) = q_{i,a}$ for each $i \in [k]$.

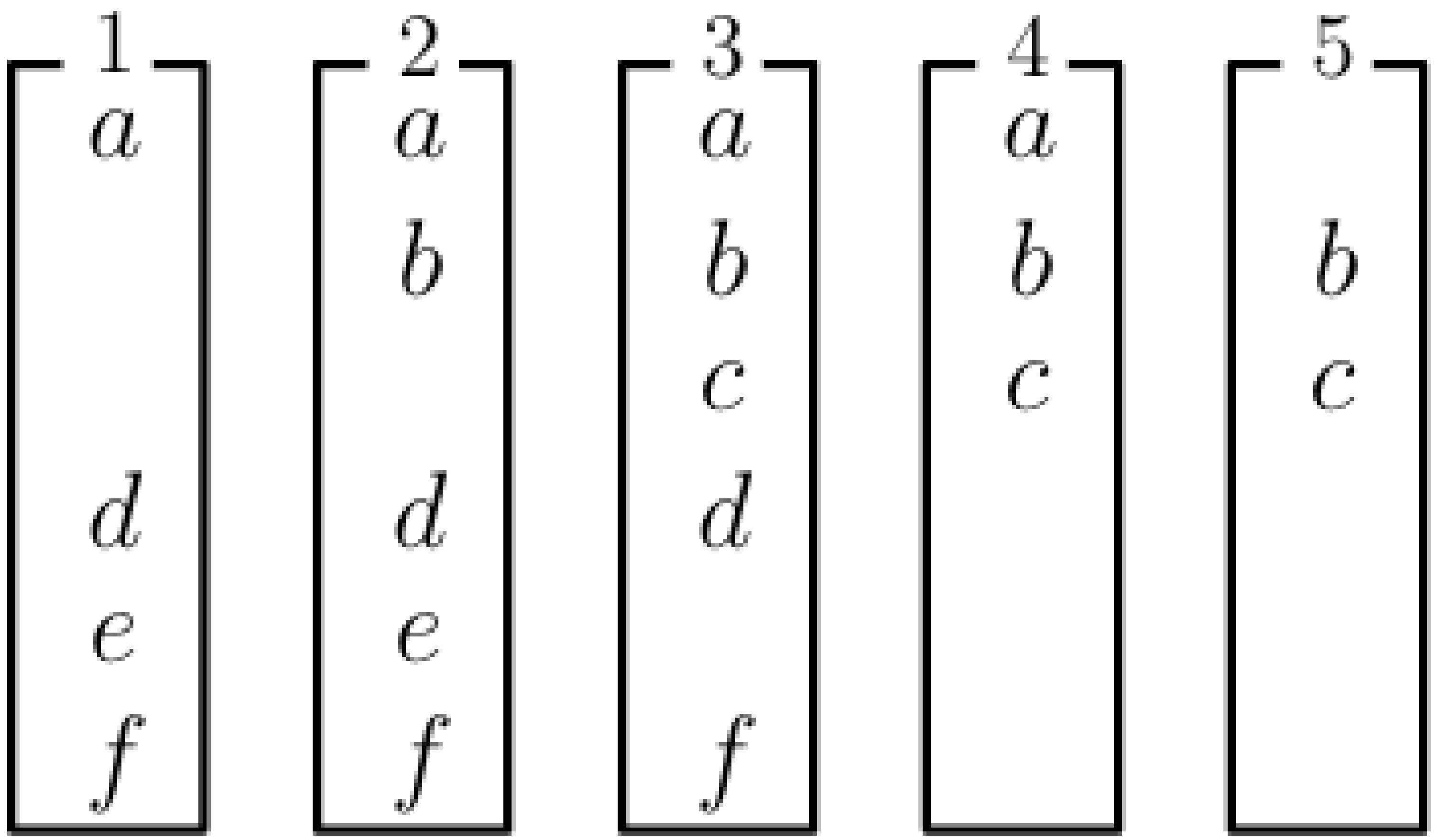




VERTEX-INTERVAL-MEMBERSHIP WIDTH

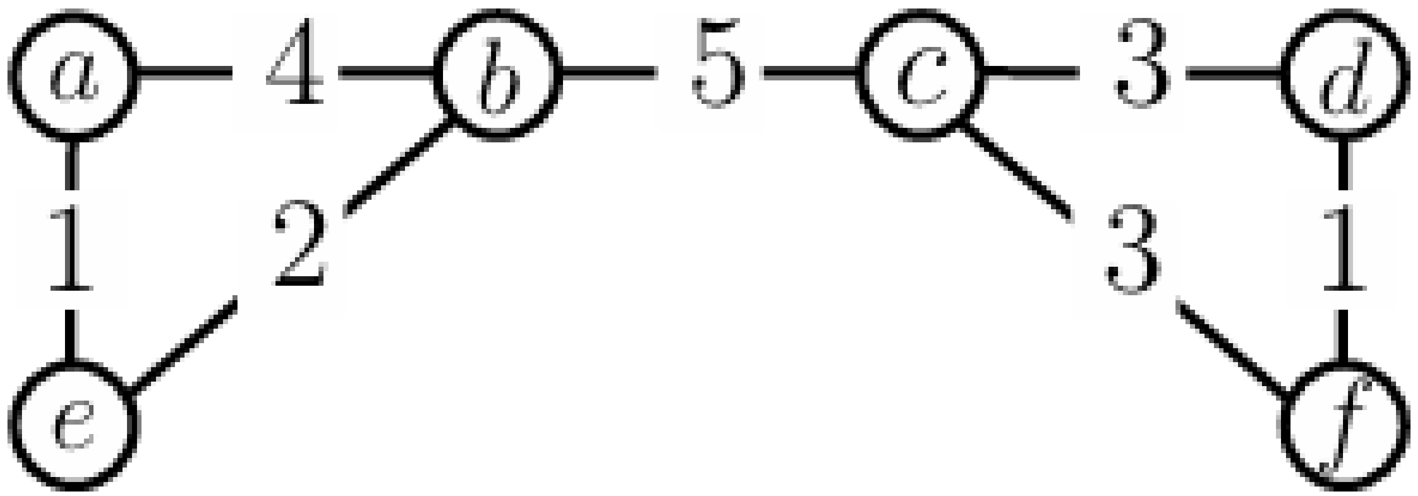
For a temporal graph $\mathcal{G}$ and a k-colour role automaton A, Temporal Role Colouring on $\mathcal{G}$ is solvable in time $\mathcal{O}(\Lambda\rho^{3\omega+1}\omega^2)$ where ω is the vertex-interval-membership width of $\mathcal{G}$ and ρ is the number of states of A.

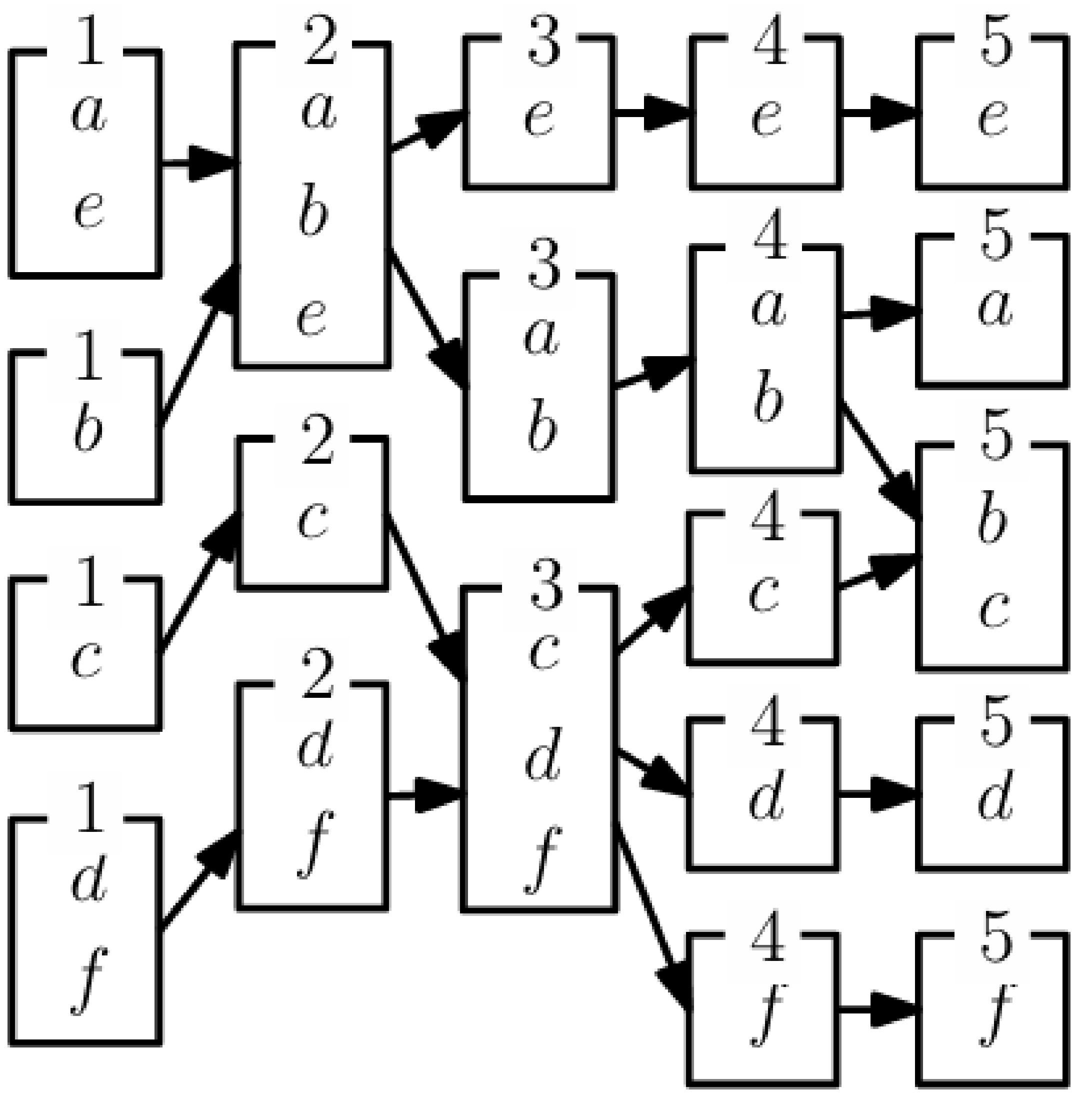




TREE-INTERVAL-MEMBERSHIP WIDTH

For a temporal graph $\mathcal{G}$ with n vertices and lifetime Λ and a k-colour role automaton A with ρ states the Temporal Role Colouring problem on $\mathcal{G}$ is solvable in time $\mathcal{O}((n\Lambda)^5(3\rho)^{12\phi^3}\rho\phi^{12})$ where ϕ is the tree-interval-membership width of $\mathcal{G}$.



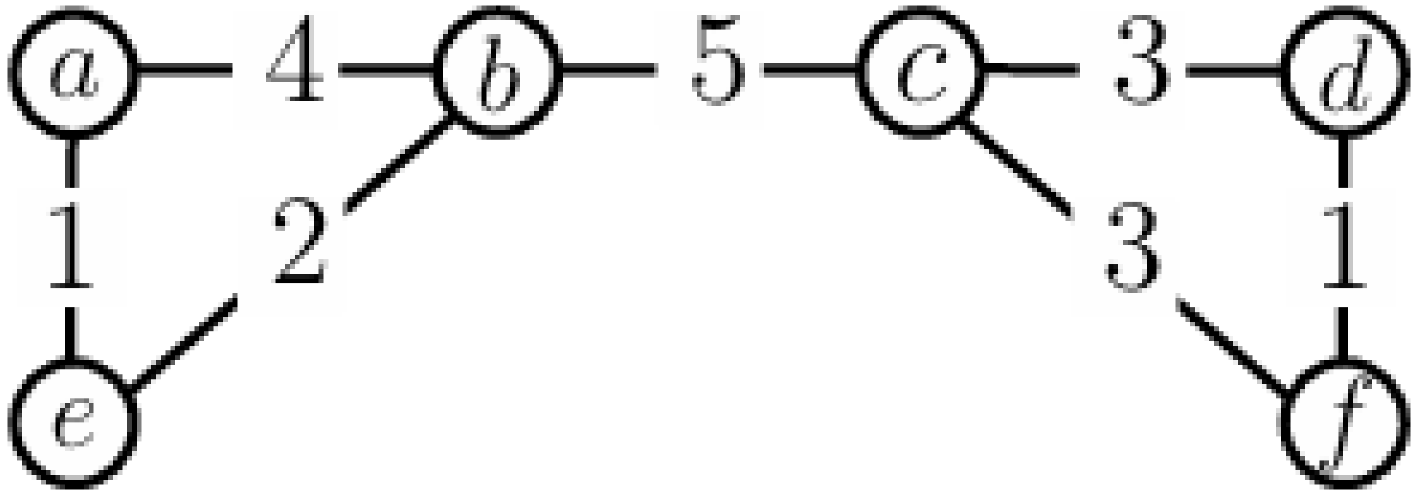


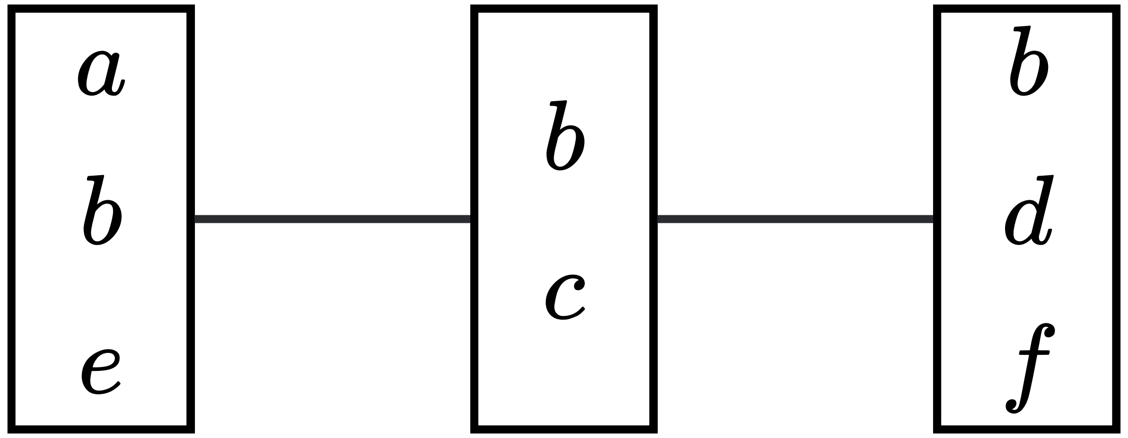
TREEWIDTH

For a temporal graph $\mathcal{G}$ and a k-colour role automaton A,

Temporal Role Colouring on instance $(\mathcal{G}, A)$ is solvable in time $\mathcal{O}(\Lambda n\omega(\rho^2 + \omega k)2^{6k(\Lambda+1)(\omega+1)})$

where ω is the treewidth of the underlying graph $G_{\downarrow}$, Λ is the lifetime of $\mathcal{G}$ and ρ is the number of states of A.





ANY QUESTIONS?