# On the Hardness of Finding Temporally Connected Subgraphs of Any Size

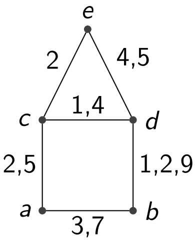
Arnaud Casteigts[1], Christian Komusiewicz[2], **Nils Morawietz**[2,3]

1: University of Geneva, Switzerland
2: Friedrich Schiller University Jena, Germany
3: Université de Bordeaux, France

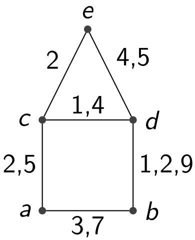
AATG IX

$\mathcal{G} = (V, E, \lambda)$, where $\lambda : E \to 2^{\mathbb{N}}$ assigns **time labels** to edges.

$\mathcal{G} = (V, E, \lambda)$, where $\lambda : E \to 2^{\mathbb{N}}$ assigns **time labels** to edges.
**footprint:** $G = (V, E)$



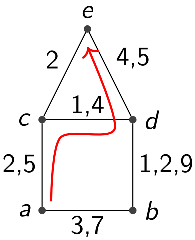
$\mathcal{G} = (V, E, \lambda)$, where $\lambda : E \to 2^{\mathbb{N}}$ assigns **time labels** to edges.
**footprint:** $G = (V, E)$




**Temporal paths**

- $\langle (a, c, 2), (c, d, 4), (d, e, 4) \rangle$ **(non-strict)**
- $\langle (a, c, 2), (c, d, 4), (d, e, 5) \rangle$ **(strict)**

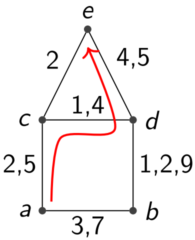
$\mathcal{G} = (V, E, \lambda)$, where $\lambda : E \to 2^{\mathbb{N}}$ assigns **time labels** to edges.
**footprint:** $G = (V, E)$




**Temporal paths**

- $\langle (a, c, 2), (c, d, 4), (d, e, 4) \rangle$ **(non-strict)**
- $\langle (a, c, 2), (c, d, 4), (d, e, 5) \rangle$ **(strict)**

$\mathcal{G} = (V, E, \lambda)$, where $\lambda : E \to 2^{\mathbb{N}}$ assigns **time labels** to edges.
**footprint:** $G = (V, E)$

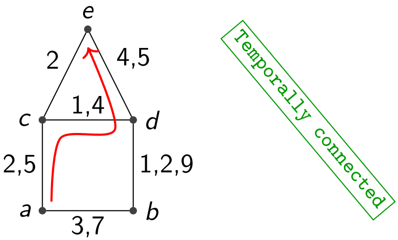



**Temporal paths**

- $\langle (a, c, 2), (c, d, 4), (d, e, 4) \rangle$ **(non-strict)**
- $\langle (a, c, 2), (c, d, 4), (d, e, 5) \rangle$ **(strict)**

**Temporal connectivity**: Temporal paths between all pairs (class TC).

$\mathcal{G} = (V, E, \lambda)$, where $\lambda : E \to 2^{\mathbb{N}}$ assigns **time labels** to edges.
**footprint:** $G = (V, E)$

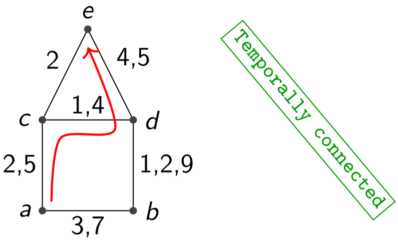



**Temporal paths**

- $\langle (a, c, 2), (c, d, 4), (d, e, 4) \rangle$ (**non-strict**)
- $\langle (a, c, 2), (c, d, 4), (d, e, 5) \rangle$ (**strict**)

**Temporal connectivity**: Temporal paths between all pairs (class TC).

$\mathcal{G} = (V, E, \lambda)$, where $\lambda : E \to 2^{\mathbb{N}}$ assigns **time labels** to edges.
**footprint:** $G = (V, E)$



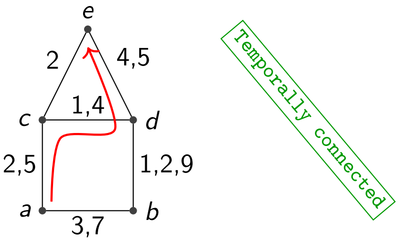

**Temporal paths**

- $\langle (a, c, 2), (c, d, 4), (d, e, 4) \rangle$ (**non-strict**)
- $\langle (a, c, 2), (c, d, 4), (d, e, 5) \rangle$ (**strict**)

**Temporal connectivity**: Temporal paths between all pairs (class TC).

$\rightarrow$ In general, reachability is non-symmetrical...

$\mathcal{G} = (V, E, \lambda)$, where $\lambda : E \to 2^{\mathbb{N}}$ assigns **time labels** to edges.
**footprint:** $G = (V, E)$



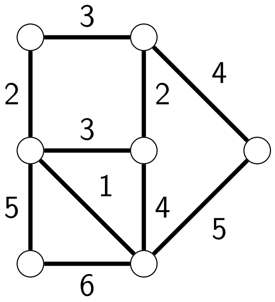

**Temporal paths**

- $\langle (a, c, 2), (c, d, 4), (d, e, 4) \rangle$ (**non-strict**)
- $\langle (a, c, 2), (c, d, 4), (d, e, 5) \rangle$ (**strict**)

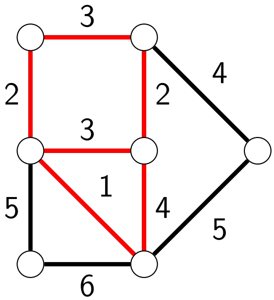
**Temporal connectivity**: Temporal paths between all pairs (class TC).

$\to$ In general, reachability is non-symmetrical... and non-transitive!

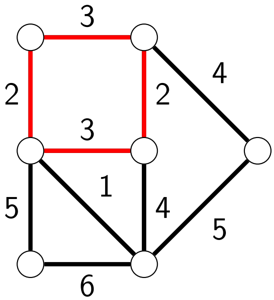
TC subgraph $\hat{=}$ a subgraph $\mathcal{G}[S]$ that is temporally connected



TC subgraph $\hat{=}$ a subgraph $\mathcal{G}[S]$ that is temporally connected

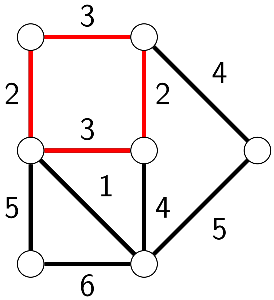


TC subgraph $\hat{=}$ a subgraph $\mathcal{G}[S]$ that is temporally connected



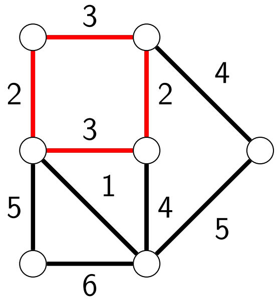
TC subgraph $\hat{=}$ a subgraph $\mathcal{G}[S]$ that is temporally connected

property is not hereditary,



TC subgraph $\hat{=}$ a subgraph $\mathcal{G}[S]$ that is temporally connected

property is not hereditary, not even quasi-hereditary




Can we decide efficiently whether there is a TC subgraph of **non-trivial** size (i.e., $> 2$)?

Can we decide efficiently whether there is a TC subgraph of **non-trivial** size (i.e., $> 2$)?

Can we decide efficiently whether there is a TC subgraph of **non-trivial** size (i.e., $> 2$)?

Related problems:

Can we decide efficiently whether there is a TC subgraph of **non-trivial** size (i.e., $> 2$)?

Related problems:

Is there a TC subgraph of size **exactly** $k$?

- NP-hard and W[1]-hard for parameter $k$ (W[1]-hardness (was) open for the non-strict case)
- XP for parameter $k$ ($n^{k+O(1)}$ time)

Can we decide efficiently whether there is a TC subgraph of **non-trivial** size (i.e., $> 2$)?

Related problems:

Is there a TC subgraph of size **exactly** $k$?

- NP-hard and W[1]-hard for parameter $k$ (W[1]-hardness (was) open for the non-strict case)
- XP for parameter $k$ ($n^{k+O(1)}$ time)

Is there a TC subgraph of size **at least** $k$?

- NP-hard and W[1]-hard for parameter $k$ (see the talk of Andrea from 2 years ago)
- No XP-algorithm is known

Can we decide efficiently whether there is a TC subgraph of **non-trivial** size (i.e., $> 2$)?

Related problems:

Is there a TC subgraph of size **exactly** $k$?

- NP-hard and W[1]-hard for parameter $k$ (W[1]-hardness (was) open for the non-strict case)
- XP for parameter $k$ ($n^{k+O(1)}$ time)

Is there a TC subgraph of size **at least** $k$?

- NP-hard and W[1]-hard for parameter $k$ (see the talk of Andrea from 2 years ago)
- No XP-algorithm is known

We consider the latter problem for the smallest value of $k$ for which it is not trivial.

Can we decide efficiently whether there is a TC subgraph of **non-trivial** size (i.e., $> 2$)?

Related problems:

Is there a TC subgraph of size **exactly** $k$?

- NP-hard and W[1]-hard for parameter $k$ (W[1]-hardness (was) open for the non-strict case)
- XP for parameter $k$ ($n^{k+O(1)}$ time)

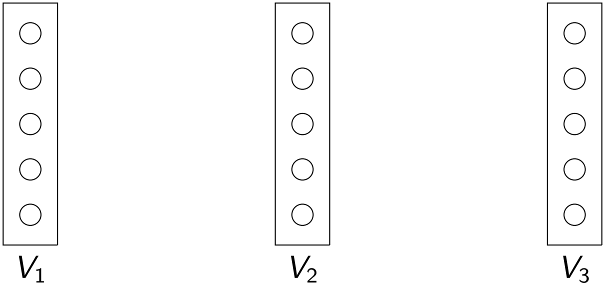
Is there a TC subgraph of size **at least** $k$?

- NP-hard and W[1]-hard for parameter $k$ (see the talk of Andrea from 2 years ago)
- No XP-algorithm is known

We consider the latter problem for the smallest value of $k$ for which it is not trivial.

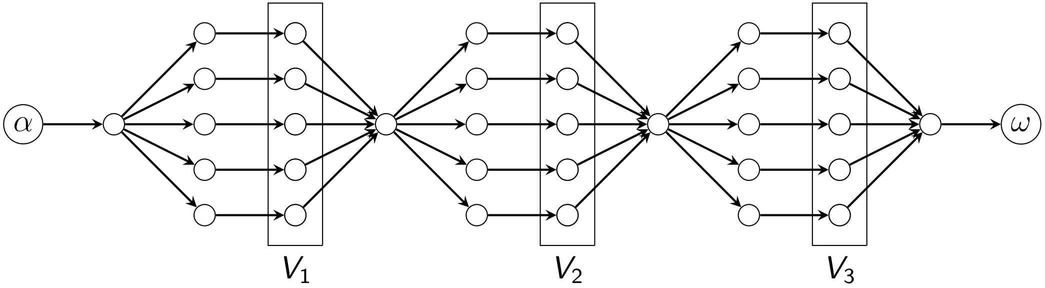
**No!**

# Reduction from Multicolored Clique



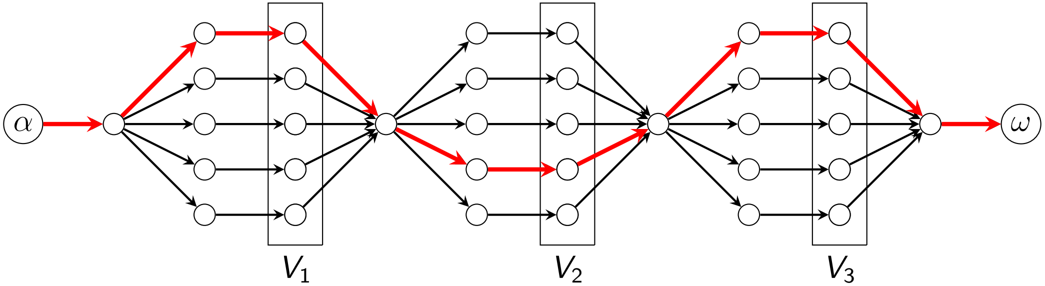
Reduction from Multicolored Clique

## Selection gadget



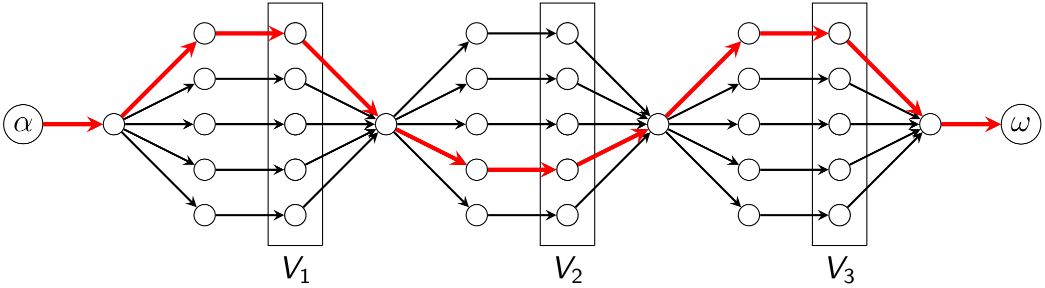
# Reduction from Multicolored Clique

## Selection gadget



Reduction from Multicolored Clique

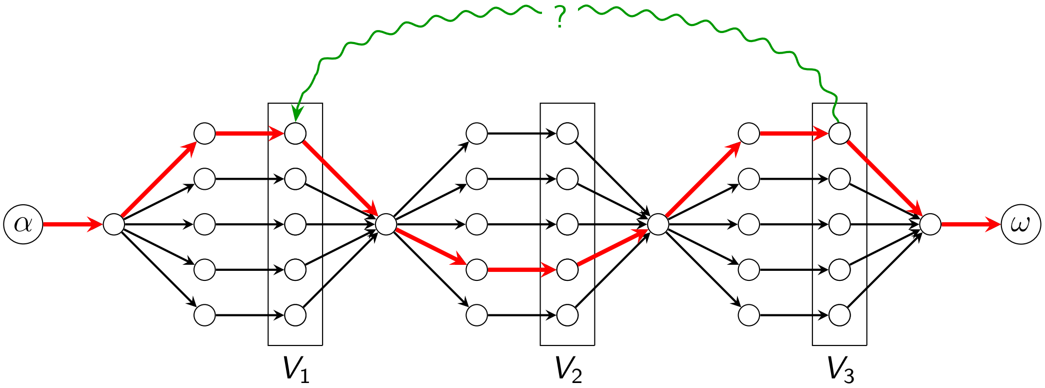
**Selection gadget**




Problem: we need to reach back!

# Reduction from Multicolored Clique

## Selection gadget

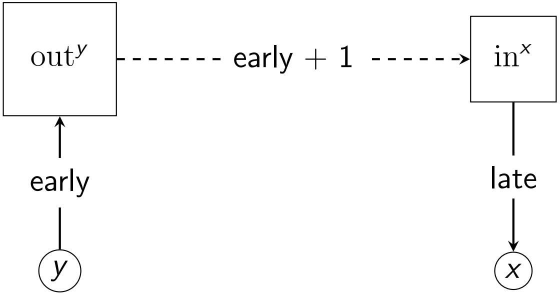



Problem: we need to reach back!

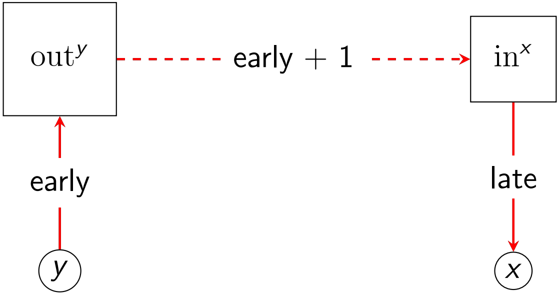
# Auxiliary vertices

# Auxiliary vertices

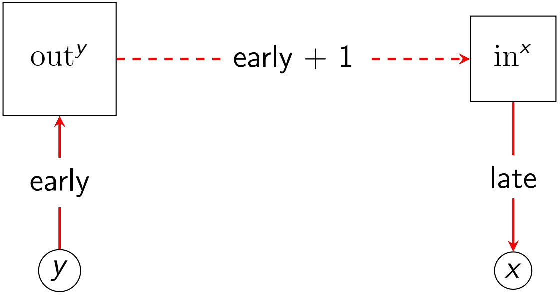
# Auxiliary vertices



## Auxiliary vertices

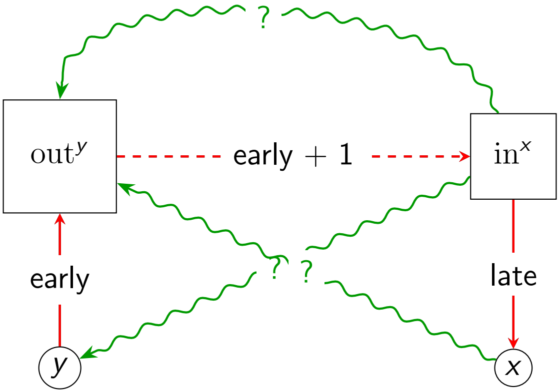


## Auxiliary vertices



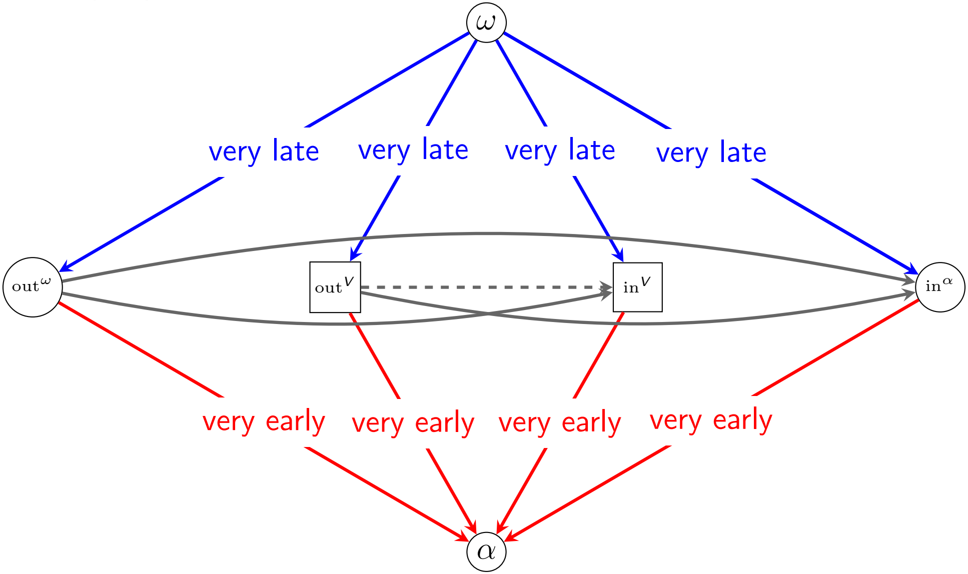

So we can safely add the auxiliary vertices?

## Auxiliary vertices

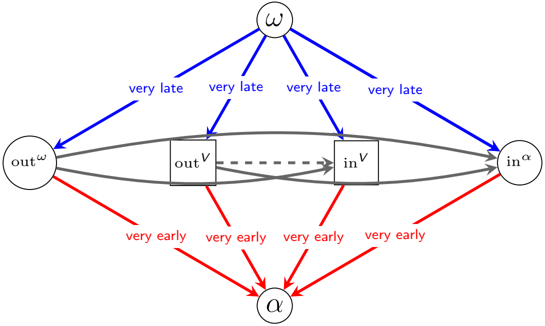


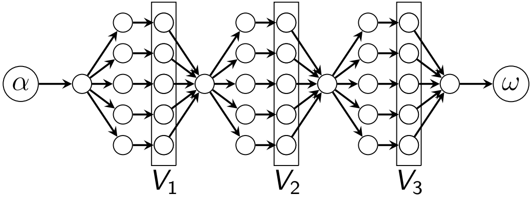

So we can safely add the auxiliary vertices? **No!**

# Connection gadget

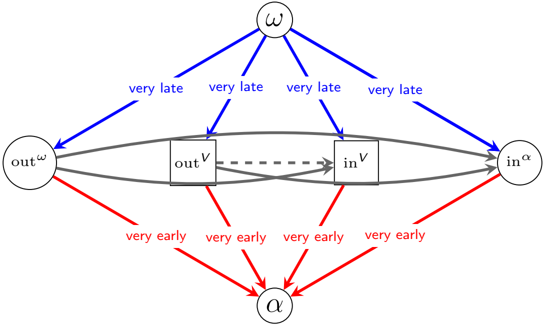


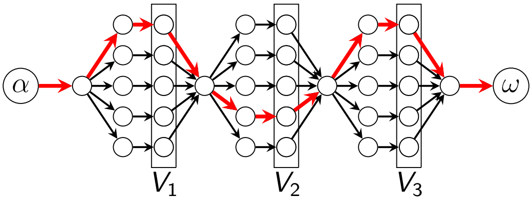
# multicolored clique ⇒ non-trivial TC subgraph



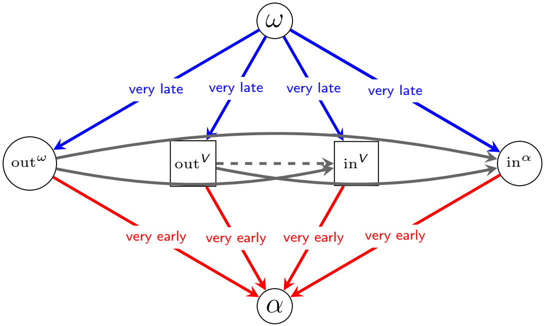
# multicolored clique ⇒ non-trivial TC subgraph

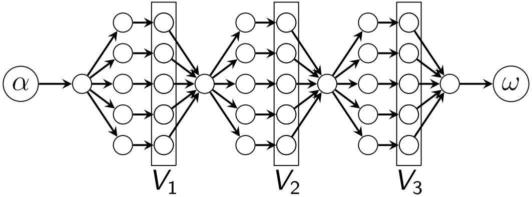


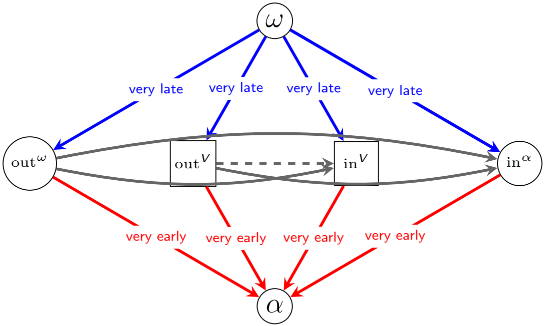

1. Let $S$ be the vertices of the $(\alpha, \omega)$-path $P$ in the selection gadget going through the multicolored clique.
2. Auxiliary vertices gather at $\alpha$ very early.
3. The vertices of $S$ can pairwise reach each other via $P$ or via out-in-paths.
4. $\omega$ can distribute to all auxiliary vertices very late.

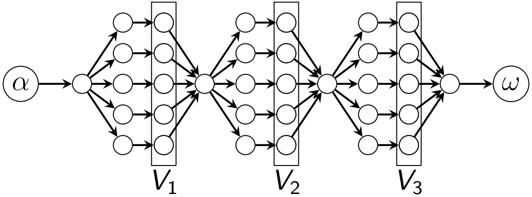
# non-trivial TC subgraph ⇒ multicolored clique



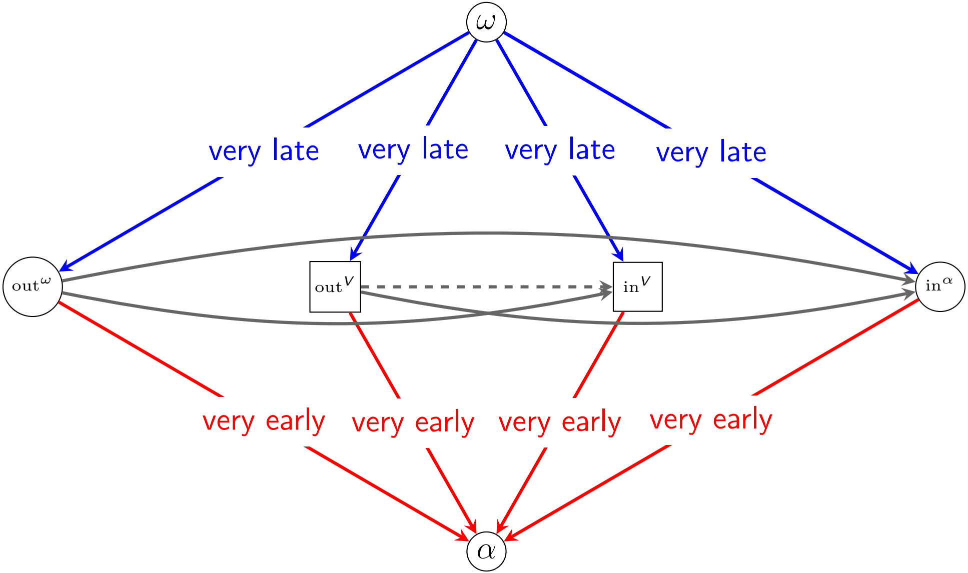
# non-trivial TC subgraph ⇒ multicolored clique




1. Every non-trivial TC subgraph contains $\alpha$ and $\omega$.
2. If a TC subgraph contains $\alpha$ and $\omega$, it contains a vertex of each color class.
3. If a TC subgraph contains two vertices $u$ and $v$ of the Clique instance, then $u$ and $v$ are neighbors (in the Clique instance).

## Problems for undirected (we have to avoid triangles in the footprint)

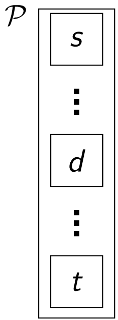


Idea: replace each auxiliary vertex by a small gadget $\mathcal{P}$

Idea: replace each auxiliary vertex by a small gadget $\mathcal{P}$ fulfilling:

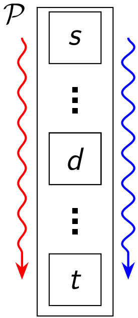
Idea: replace each auxiliary vertex by a small gadget $\mathcal{P}$ fulfilling:

- $\mathcal{P}$ contains no non-trivial TC subgraph,



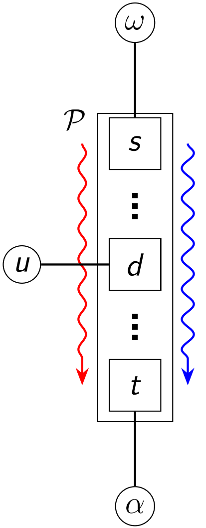
Idea: replace each auxiliary vertex by a small gadget $\mathcal{P}$ fulfilling:

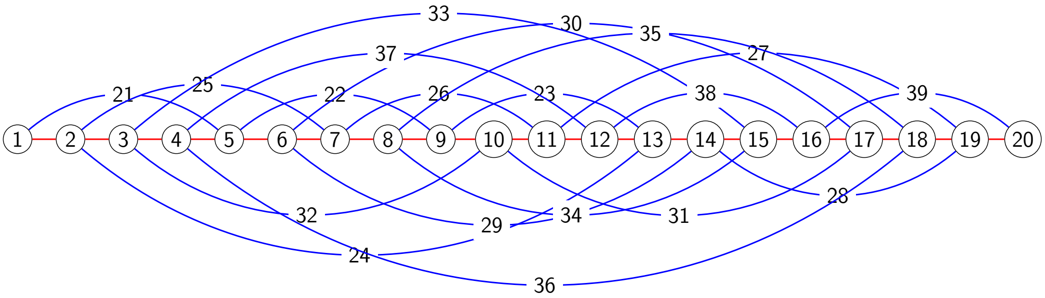
- $\mathcal{P}$ contains no non-trivial TC subgraph,
- $\mathcal{P}$ has two edge-disjoint spanning trees (to gather early and distribute late), and



Idea: replace each auxiliary vertex by a small gadget $\mathcal{P}$ fulfilling:

- $\mathcal{P}$ contains no non-trivial TC subgraph,
- $\mathcal{P}$ has two edge-disjoint spanning trees (to gather early and distribute late), and
- there is a vertex ('docking point') in $\mathcal{P}$ that can be attached to vertices of the selection gadget and docking points of other gadgets $\mathcal{P}'$ without creating small non-trivial TC subgraphs.






red path = (1,2,...,20) $\quad \forall i \in [1, 19]$ : edge $\{i, i+1\}$ has label $i$

blue path = (1,5,9,13,2,7,11,19,14,6,17,10,3,15,8,18,4,12,16,20)

### Lemma

*A temporally connected graph has at least $2n - 3$ edges if its footprint has girth is at least 5.*

**Lemma**

*A temporally connected graph has at least $2n - 3$ edges if its footprint has girth is at least 5.*

**Lemma**

*A graph of girth is at least 5 with at least 3 vertices and at least $2n - 3$ edges fulfills $n \geq 17$.*

**Lemma**

*A temporally connected graph has at least $2n-3$ edges if its footprint has girth is at least 5.*

**Lemma**

*A graph of girth is at least 5 with at least 3 vertices and at least $2n-3$ edges fulfills $n \geq 17$.*

**Lemma**

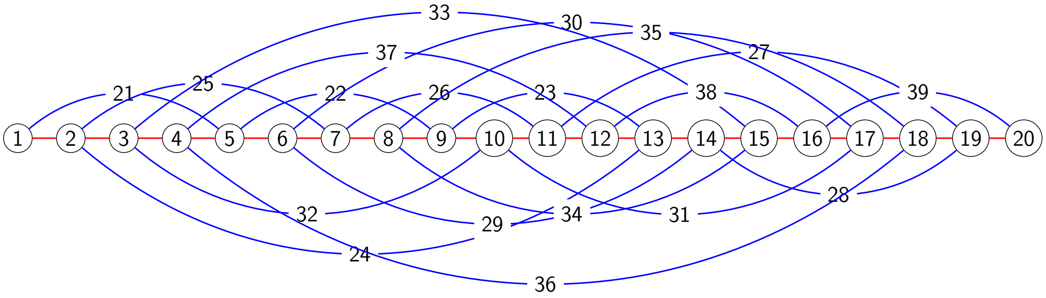
*A temporally connected graph of non-trivial size with footprint of girth at least 5 has at least 17 vertices.*

## Lemma

*A temporally connected graph of non-trivial size with footprint of girth at least 5 has at least 17 vertices.*

## Lemma

*A temporally connected graph of non-trivial size with footprint of girth at least 5 has at least 17 vertices.*

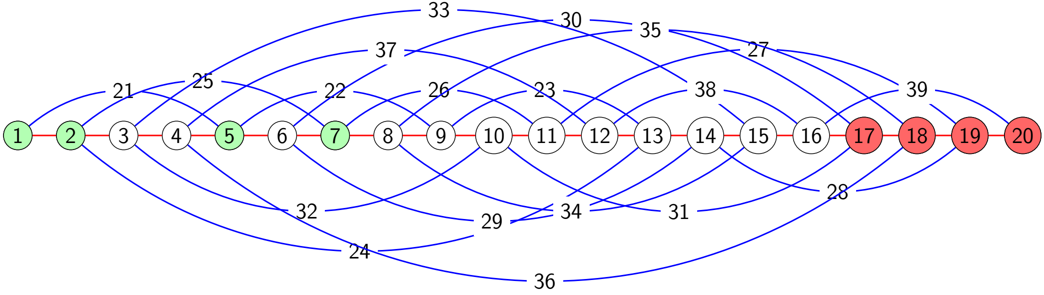



red path $=$ (1,2,...,20) $\quad \forall i \in [1, 19]$ : edge $\{i, i+1\}$ has label $i$

blue path $=$ (1,5,9,13,2,7,11,19,14,6,17,10,3,15,8,18,4,12,16,20)

## Lemma

*A temporally connected graph of non-trivial size with footprint of girth at least 5 has at least 17 vertices.*

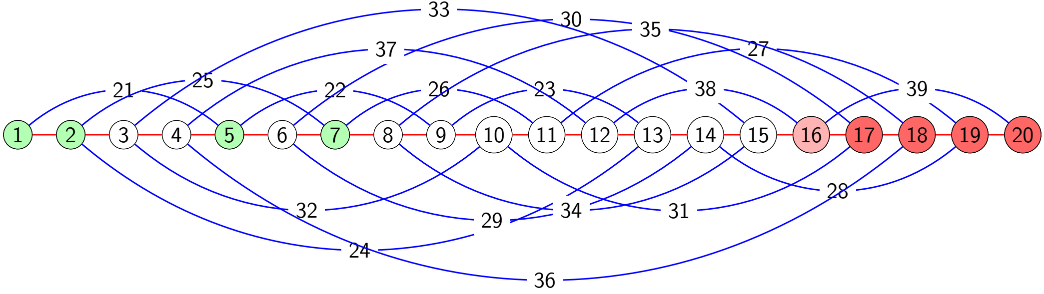



red path $= (1,2,...,20)$ $\quad \forall i \in [1, 19]$ : edge $\{i, i+1\}$ has label $i$

blue path $= (1,5,9,13,2,7,11,19,14,6,17,10,3,15,8,18,4,12,16,20)$

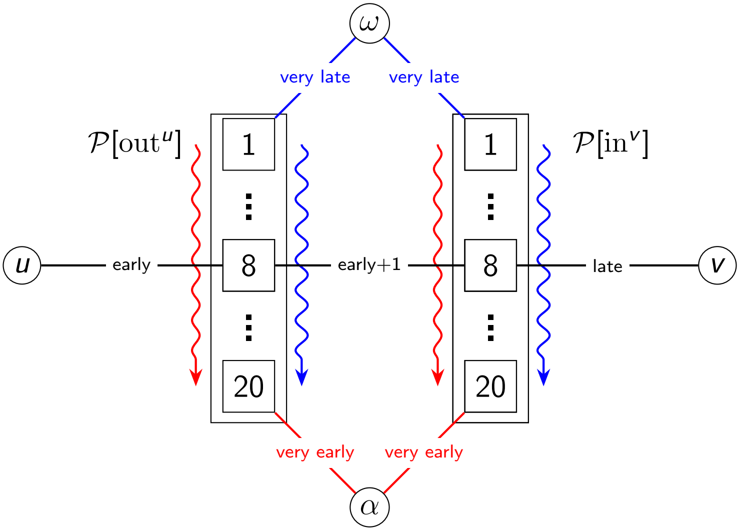
## Lemma

*A temporally connected graph of non-trivial size with footprint of girth at least 5 has at least 17 vertices.*




red path = (1,2,...,20) $\quad \forall i \in [1, 19]$ : edge $\{i, i+1\}$ has label $i$

blue path = (1,5,9,13,2,7,11,19,14,6,17,10,3,15,8,18,4,12,16,20)



$\omega$
very late
very late
$\mathcal{P}[\mathrm{out}^u]$
$\mathcal{P}[\mathrm{in}^v]$
1
1
8
8
20
20
$u$
early
early+1
late
$v$
very early
very early
$\alpha$

Intractability implications:

- ETH lower bounds (no $2^{o(n)}$ in general graphs and no $2^{o(\sqrt{n})}$ in the undirected happy case)

Intractability implications:

- ETH lower bounds (no $2^{o(n)}$ in general graphs and no $2^{o(\sqrt{n})}$ in the undirected happy case)
- Deciding wether the TC subgraph consisting of an edge (vertex) in the undirected (directed) case is hard

Intractability implications:

- ETH lower bounds (no $2^{o(n)}$ in general graphs and no $2^{o(\sqrt{n})}$ in the undirected happy case)
- Deciding wether the TC subgraph consisting of an edge (vertex) in the undirected (directed) case is hard
- No approximation better than $(1-\varepsilon)\frac{n}{2}$ ($(1-\varepsilon)n$ for directed) for the size of the largest TC subgraph

Intractability implications:

- ETH lower bounds (no $2^{o(n)}$ in general graphs and no $2^{o(\sqrt{n})}$ in the undirected happy case)
- Deciding wether the TC subgraph consisting of an edge (vertex) in the undirected (directed) case is hard
- No approximation better than $(1-\varepsilon)\frac{n}{2}$ ($(1-\varepsilon)n$ for directed) for the size of the largest TC subgraph
- Is there a TC subgraph of size exactly $k$ in happy graphs is $W[1]$-hard

Intractability implications:

- ETH lower bounds (no $2^{o(n)}$ in general graphs and no $2^{o(\sqrt{n})}$ in the undirected happy case)
- Deciding wether the TC subgraph consisting of an edge (vertex) in the undirected (directed) case is hard
- No approximation better than $(1-\varepsilon)\frac{n}{2}$ ($(1-\varepsilon)n$ for directed) for the size of the largest TC subgraph
- Is there a TC subgraph of size exactly $k$ in happy graphs is $W[1]$-hard

Intractability implications:

- ETH lower bounds (no $2^{o(n)}$ in general graphs and no $2^{o(\sqrt{n})}$ in the undirected happy case)
- Deciding wether the TC subgraph consisting of an edge (vertex) in the undirected (directed) case is hard
- No approximation better than $(1-\varepsilon)\frac{n}{2}$ ($(1-\varepsilon)n$ for directed) for the size of the largest TC subgraph
- Is there a TC subgraph of size exactly $k$ in happy graphs is $W[1]$-hard

Structural implications:

- There are TC graphs of arbitrary size that contain no non-trivial TC subgraph

Intractability implications:

- ETH lower bounds (no $2^{o(n)}$ in general graphs and no $2^{o(\sqrt{n})}$ in the undirected happy case)
- Deciding wether the TC subgraph consisting of an edge (vertex) in the undirected (directed) case is hard
- No approximation better than $(1-\varepsilon)\frac{n}{2}$ ($(1-\varepsilon)n$ for directed) for the size of the largest TC subgraph
- Is there a TC subgraph of size exactly $k$ in happy graphs is $W[1]$-hard

Structural implications:

- There are TC graphs of arbitrary size that contain no non-trivial TC subgraph
- For any $g \geq 3$, there are TC graphs of girth at least $g$

Intractability implications:

- ETH lower bounds (no $2^{o(n)}$ in general graphs and no $2^{o(\sqrt{n})}$ in the undirected happy case)
- Deciding wether the TC subgraph consisting of an edge (vertex) in the undirected (directed) case is hard
- No approximation better than $(1-\varepsilon)\frac{n}{2}$ ($(1-\varepsilon)n$ for directed) for the size of the largest TC subgraph
- Is there a TC subgraph of size exactly $k$ in happy graphs is $W[1]$-hard

Structural implications:

- There are TC graphs of arbitrary size that contain no non-trivial TC subgraph
- For any $g \geq 3$, there are TC graphs of girth at least $g$

Questions:

Questions:

What about the parameterized complexity of the problem wrt. treewidth, ...?

Questions:

What about the parameterized complexity of the problem wrt. treewidth, ...?

Can we solve the problems for non-strict paths in $2^{\mathcal{O}(\sqrt{n})}$ time?

Questions:

What about the parameterized complexity of the problem wrt. treewidth, ...?

Can we solve the problems for non-strict paths in $2^{\mathcal{O}(\sqrt{n})}$ time?

What if the input is already TC and we just ask for a non-trivial TC subgraph which misses at least one vertex?