Minimum Temporal Spanners in Happy Graphs

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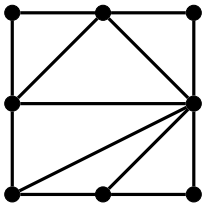
Hasso Plattner Institute, Potsdam, Germany

Algorithmic Aspects of Temporal Graphs IX

Based on joint work with Arnaud Casteigts and Meirav Zehavi.

Motivation

Spanning Trees



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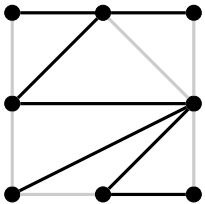

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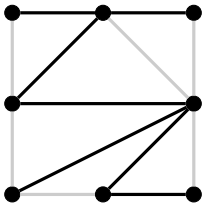

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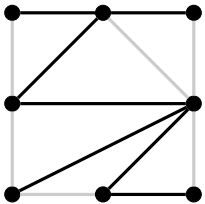

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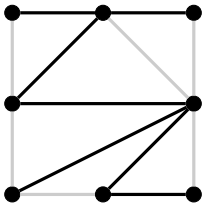

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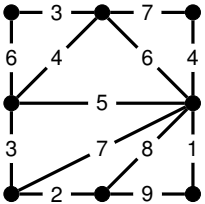

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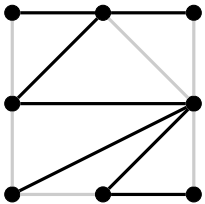

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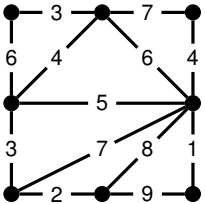

Canonical generalization to the temporal setting.

Temporal Graphs and Temporal Connectivity: Notation and Definition

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$\mathscr{G} = (V, (E_i)_{i \in [T]})$: vertex set V, edge sets $E_1, \ldots, E_T$ over V, where T is lifetime of $\mathscr{G}$.

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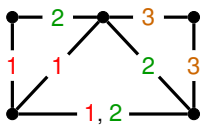

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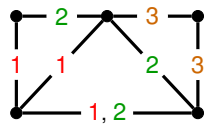

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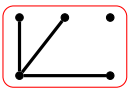

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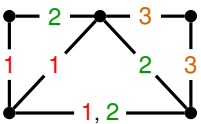

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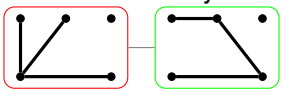

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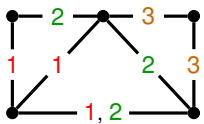

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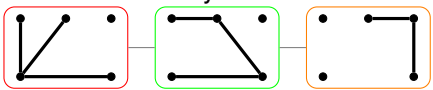

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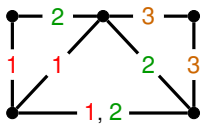

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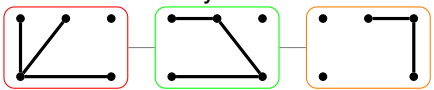
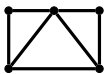


Underlying graph:



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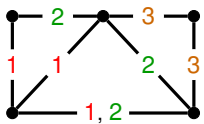

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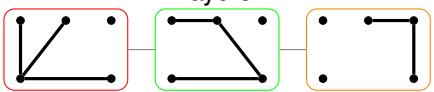

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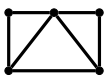

Temporal (s, z)-Path

Sequence of time edges forming a path from s to z that have:

- increasing time stamps (strict).
- non-decreasing time stamps (non-strict).

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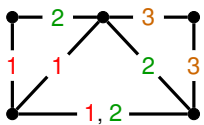

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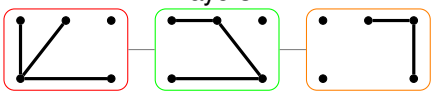

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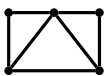

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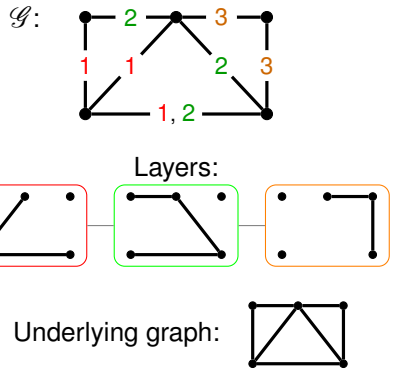

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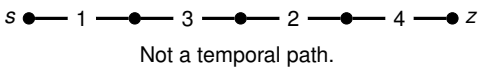

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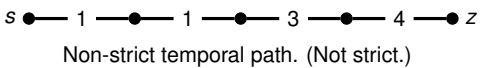

Non-strict temporal path. (Not strict.)

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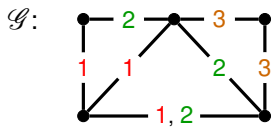

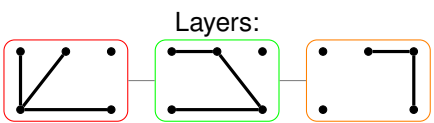

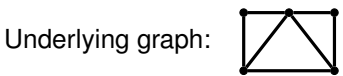

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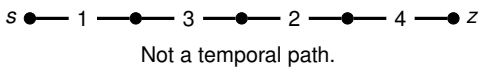

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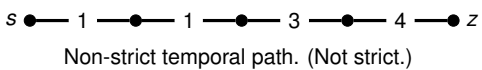

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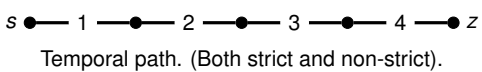

Temporal path. (Both strict and non-strict).

Oddities of Temporal Connectivity

Connectivity in the static (undirected) setting: symmetric and transitive.

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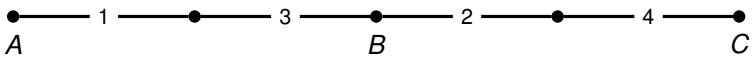

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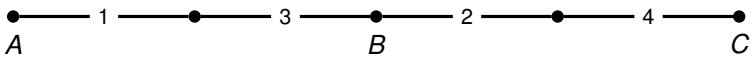

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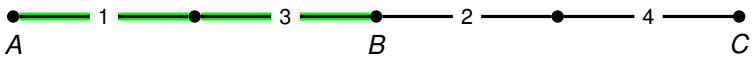

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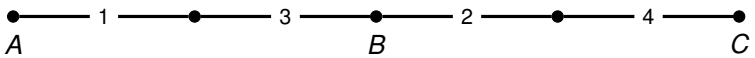

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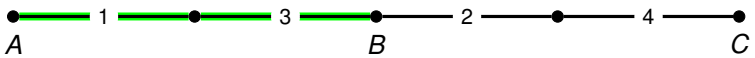

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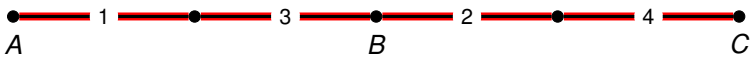

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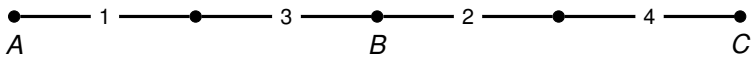

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Observation

Connectivity-related problems are computationally much harder in the temporal setting.

Minimum Temporal Spanners: Definition

Definition

A temporal graph $\mathscr{G}$ is (strictly or non-strictly) **temporally connected** (or **TC**) if there is a (strict or non-strict) temporal path between each pair of vertices (in both directions).

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Input: A TC temporal graph $\mathscr{G}$ and an integer k.
Task: Compute a TC spanning subgraph of $\mathscr{G}$ of size k (or decide that none exists).

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4 Problem Variants: strict, non-strict $\times$ min label, min edge

Simple, Proper, and Happy Temporal Graphs

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Only one label per edge!

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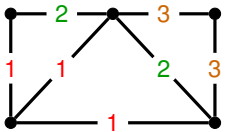

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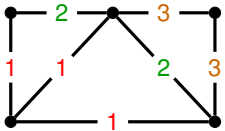

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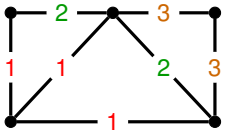

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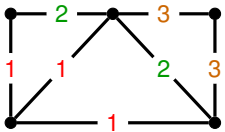
Proper Temporal Graphs

Different labels around each vertex!

Simple, Proper, and Happy Temporal Graphs

Simple Temporal Graphs

Only one label per edge!



min label = min edge

Proper Temporal Graphs

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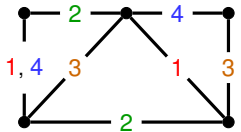

Simple, Proper, and Happy Temporal Graphs

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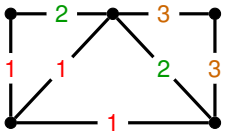

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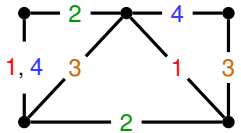

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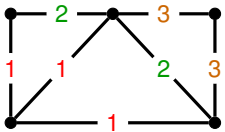

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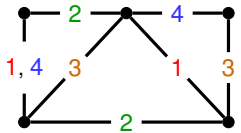

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Happy Temporal Graphs: Simple and proper!

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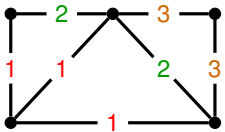

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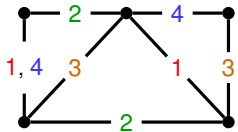

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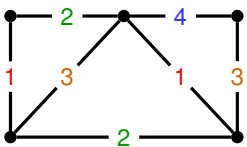

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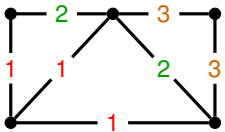

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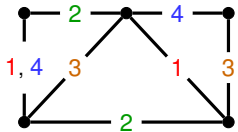

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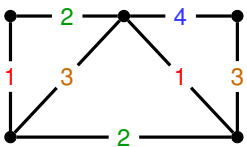

All 4 problem variants are the same!

Known Results

NP-Hardness Results

Article	Setting	str / non-str	min label / edge
Axiotis and Fotakis [ICALP '16]	simple & not proper	non-strict	both
Akrida et al. [TOCS '17]	not simple & not proper	strict	min label
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Algorithmic Results

Axiotis and Fotakis [ICALP '16] show poly-time solvability if underlying graph is a tree.

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Long line of research on structural properties of temporal spanners (especially in so-called temporal cliques).

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- Kurita et al. [SAND '25] and Cioni et al. [IWOCA '26] study the enumeration of temporal spanners.

Our Results

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Theorem

Minimum Temporal Spanner (min label) is W[1]-hard when parameterized by the feedback vertex number of the underlying graph on proper temporal graphs.

XP Algorithm for Vertex Cover Number I

Definition

A **temporal out-tree** T_v rooted at v is a minimal temporal graph in which there is a temporal path from v to each other vertex.

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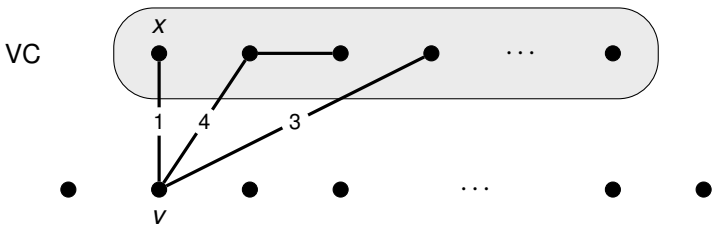

Let $v \in V_x$ for $x \in$ VC if v's time edge to x has the smallest label among time edges incident with v.

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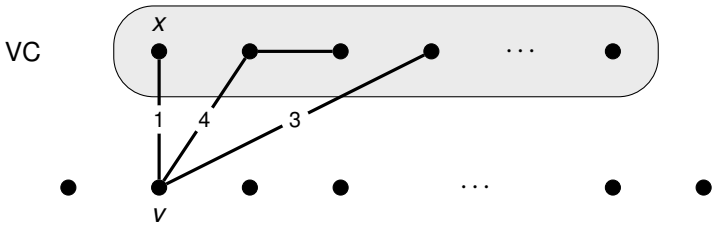
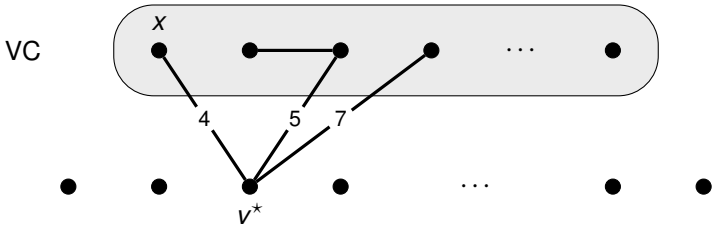


Let $v \in V_x$ for $x \in$ VC if v's time edge to x has the smallest label among time edges incident with v.

This forms a partition of the vertice in the independent set.

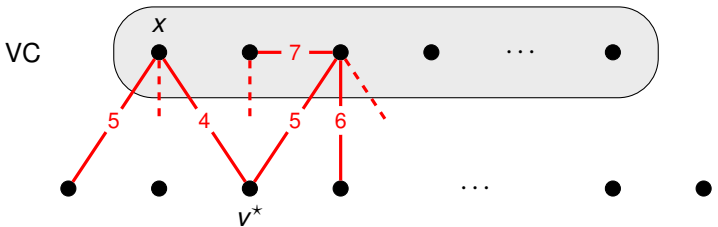
Proof of VC-Tree Lemma II



Let $v^\star \in V_x$ the vertex in V_x that has the largest label on its time edge to x among time edges between vertices in V_x and x.

Proof of VC-Tree Lemma III

Let $T_{v^\star}$ be the temporal out-tree through which $v^\star$ reaches all other vertices.



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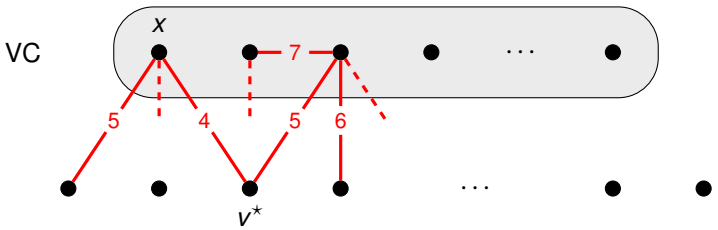

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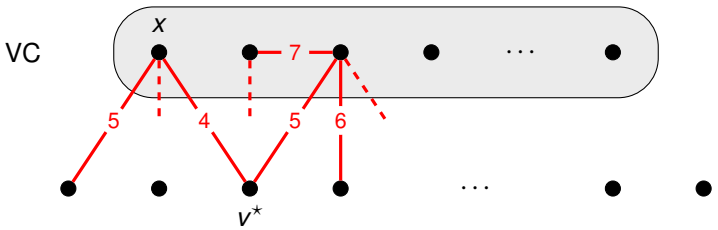

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Each vertex in V_x can reach all other vertice by first reaching x via the direct edge, and then following T_x.

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- We verify that the spanner is temporally connected and check whether its size is at most k.

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