Structural thresholds in doubly random temporal graphs

Based on joint work with George B. Mertzios and Paul Spirakis

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What do we mean by a (Random) Temporal Graph?

Temporal Graph

A **temporal graph** $\mathcal{G} = (V, E, \lambda)$ is a collection of vertices V, edges between them $E \subset V^2$ and an assignment of **time labels** to those edges $\lambda : E \to 2^{\mathbb{R}}$. We refer to $G = (V, E)$ as **the footprint** and $\max_{e \in E}(\lambda(e))$ as **the lifetime**.

We interpret $t \in \lambda(e)$ as the edge e existing at time t. We are principally interested in **random temporal graphs**, i.e. a sample from some specific distribution of temporal graphs, e.g.

- Edge-Markovian Temporal Graphs
- Random Simple Temporal Graphs
- Random Spanning Tree Model

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- Edge-Markovian Temporal Graphs
- Random Simple Temporal Graphs
- Random Spanning Tree Model
- Doubly Random Temporal Graphs (DRTG)

The Doubly Random Model

The DRTG is parametrised by the **label distribution controlling the number of labels on each edge**.

Label distribution

A label distribution is **an arbitrary discrete distribution** supported only on the non-negative integers with a finite non-zero second moment.

Doubly Random Temporal Graph

We define the distribution $\Gamma_V(\psi)$ of random temporal graphs on the vertex set V with label distribution ψ. A sample $\mathcal{G} = (V, E, \lambda)$ from $\Gamma_V(\psi)$ is obtained as follows: For each edge $e \in \binom{V}{2}$, we sample **the number of labels independently and identically from** ψ and **the value of each edge label uniformly at random**.

The Doubly Random Model(s) II

Continuous Doubly Random Temporal Graph

We define the distribution $\Gamma_V(\psi)$ of random temporal graphs on the vertex set V with label distribution ψ. A sample $\mathcal{G} = (V, E, \lambda)$ from $\Gamma_V(\psi)$ is obtained as follows: For each edge $e \in \binom{V}{2}$, we sample the number of labels independently and identically from ψ and **the value of each edge label uniformly at random from** $(\mathbf{0}, \mathbf{1}]$.

Discrete Doubly Random Temporal Graph

We define the distribution $\Gamma_V(\psi, T)$ of random temporal graphs on the vertex set V with label distribution ψ. A sample $\mathcal{G} = (V, E, \lambda)$ from $\Gamma_V(\psi)$ is obtained as follows: For each edge $e \in \binom{V}{2}$, we sample the number of labels independently and identically from ψ and **the value of each edge label uniformly at random from** $\{\mathbf{1}, ..., \mathbf{T}\}$.

What is a temporal motif?

Temporal motifs are an alternative to temporal subgraphs popular in network science.

Temporal Motif

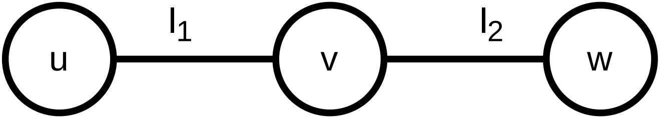
A temporal motif consists of a static graph $H = (V_H, E_H)$ and a partial order over a set edge labels on E. A temporal graph $\mathcal{G}$ contains (H, P) if there exists a temporal sub-graph $\mathcal{I} = (V_I, E_I, \lambda_I)$ of $\mathcal{G}$, such that I has footprint H and the labels of λ_I saturate and obey P.

For example, the following can all be captured as motifs:

- Fixed length time respecting paths,
- Fixed length time respecting cycles,
- Repeated edges,
- Subgraphs of the footprint.

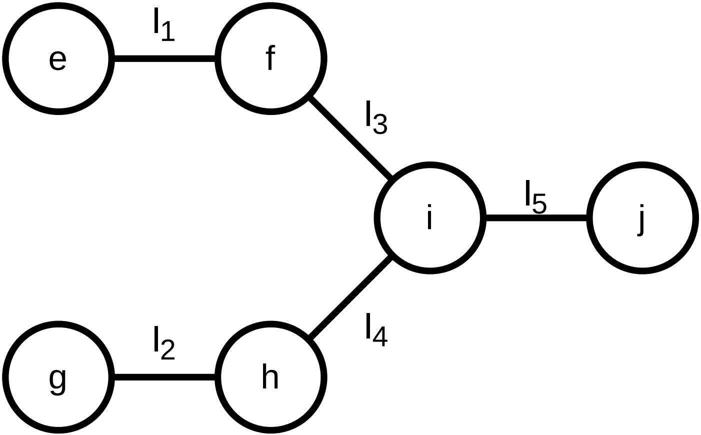
Forbidding motifs can even capture notions such as simplicity and properness.

Example of temporal motifs: Time respecting path



$l_1 < l_2$

Example of temporal motifs: Converging paths



$l_1 < l_3 < l_5$
$l_2 < l_4 < l_5$

What is a δ temporal motif?

A δ-temporal motif is a specialisation of temporal motif.

δ-Temporal Motif

A temporal graph $\mathcal{G}$ contains (H, P) if there exists a temporal sub-graph $\mathcal{I} = (V_I, E_I, \lambda_I)$ of $\mathcal{G}$, such that I has footprint H, the labels of λ_I saturate and obey P, **and all fall within an interval of length** δ.

The Fraud Detection Problem

We are given **a temporal network of transactions** and wish to identify **any cyclic sequences of transactions** that occur **within a suspiciously short window of time**.

Issue: What qualifies as a suspiciously short window of time?

Threshold for constant size motifs

A., Mertzios, Spirakis 2026

Let (H, P) be a fixed, strict and simple motif. Then, for any valid label distribution ψ, a sample from $\Gamma_{[n]}(\psi)$ contains (H, P) as a $\delta(n)$ temporal motif with high probability if $\delta = \omega(n^{-\rho(H)})$ and with high probability does not contain (H, P) a $\delta(n)$-temporal motif if $\delta = o(n^{-\rho(H)})$.

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Sparsity

For a graph $G = (V, E)$, the *sparsity* ρ_G of G is defined as follows,

$$\rho_G = \min_{H \sqsubseteq G} \frac{|V_H|}{|E_H| - 1},$$

where $H \sqsubseteq G$ denotes that H is a subgraph of G containing at least 2 edges and V_H (resp. E_H) are the set of vertices (resp. edges) comprising H.

Threshold for constant size motifs II

Bollobas 1981 (Adapted from Frieze 2026)

Let H be a static graph. Then, a sample from $G(n, \delta(n))$ contains H as a subgraph if $\delta = \omega(n^{-\sigma(H)})$ and with high probability does not contain H as a subgraph if $\delta = o(n^{-\sigma(H)})$.

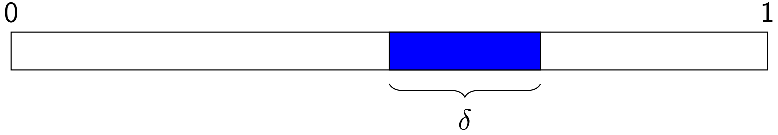
Minimal Reciprocal Density

For a graph $G = (V, E)$, the *minimal reciprocal density* σ_G of G is defined as follows,

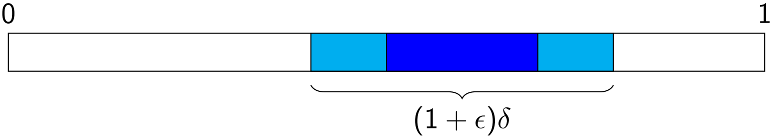
$$\sigma_G = \min_{H \subseteq G} \frac{|V_H|}{|E_H|},$$

where $H \subseteq G$ denotes that H is a subgraph of G and V_H (resp. E_H) are the set of vertices (resp. edges) comprising H.

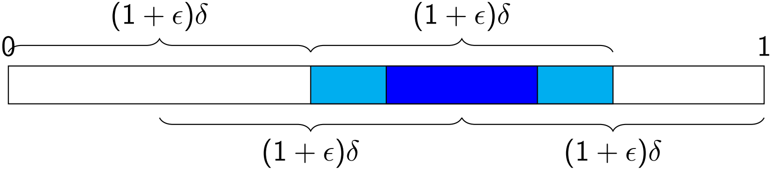
Threshold for constant size motifs IIIa



Threshold for constant size motifs IIIb



Threshold for constant size motifs IIIc



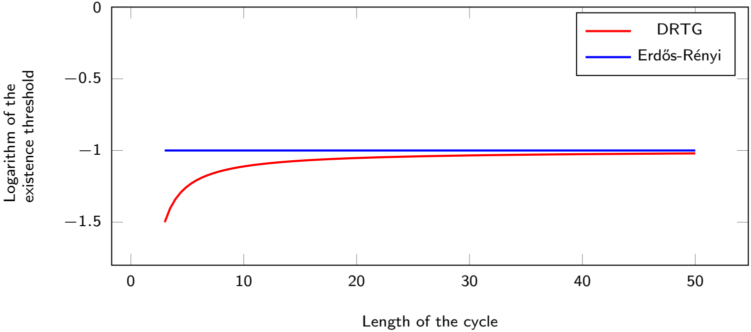
We need $\Theta(\frac{1}{\delta})$ sub-intervals to cover the whole intervals, contributing an extra factor of $\frac{1}{\delta}$ in the calculations.

So are they distinct at all?

In the $G(n, p)$ all cycles have the same asymptotic existence threshold, but...

Existence threshold for cyclic motifs

The threshold for any fixed, strict, simple motif with a C_k footprint has an existence threshold of $n^{-\frac{k}{k-1}}$.



Take away: If you want to launder money quickly, fewer hops is better.

Doubling time

We want to establish **how well connected** the DRTG is.

Doubling time

The doubling time of a set S is the minimum time step such that $2|S|$ vertices can be reached from S. The doubling time **Double**$(\mathcal{G})$ of a temporal graph $\mathcal{G}$ is the maximum doubling time over all sets of size at most $\frac{n}{2}$.

Threshold for doubling time

One can obtain the following,

A., Mertzios, Spirakis 2026

Let $\mathcal{G}$ be a sample from $\Gamma_{[n]}(\chi)$ for χ a valid degenerate label distribution with expectation x. Then for any $\epsilon > 0$, with high probability,

$$\frac{(2-\epsilon)\log n}{xn} \leq \textbf{Double}(\mathcal{G}) \leq \frac{(2+\epsilon)\log n}{xn}$$

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$$\frac{(2-\epsilon)\log n}{xn} \leq \mathbf{Double}(\mathcal{G}) \leq \frac{(2+\epsilon)\log n}{xn}$$

Corollary

Let $\mathcal{H}$ be a sample from the Random Simple Temporal Graph with parameter $\beta(n)$. Then the doubling time of $\mathcal{H}$ is finite with high probability if $\beta(n) > \frac{(2+\Omega(1))\log n}{n}$ and with high probability infinite if $\beta(n) < \frac{(2-\Omega(1))\log n}{n}$.

Map of the analysis

Discard the temporal information and consider only the order of edges.
Bound the time taken for any given set to grow across a sequence of cuts.

Upper bound

Bound the time for most vertices to be reached by $\frac{3n+1}{4}$ vertices.

Bound the time for these vertices to reach the rest.

Exploit the time reversibility of the distribution to get the doubling time.

Lower bound

Show that most sets are blocked from doubling too quickly by at least one vertex.

Show that $\Omega(n)$ vertices block at least one set each.

Show that some blocker vertices are not included in an edge for a long time and so delay the doubling time.

Convert back to the DRTG by reintroducing the temporal information.

Open Problems/Future work

- Replacing the uniform distribution for the timing of labels
- Relaxing the strictness and simpleness condition for motifs
- Growing motifs
- Non-degenerate and discrete doubling times
- Temporal Expansion

Thank You!

Any questions?