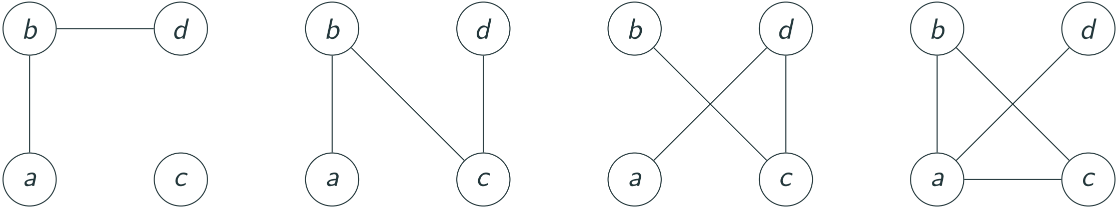
How to Color Temporal Graphs to Ensure Proper Transitions

Allen Ibiapina, Minh-Hang Nguyen, Mikaël Rabie, Cléophée Robin

Algorithmic Aspects of Temporal Graphs - ICALP 2025

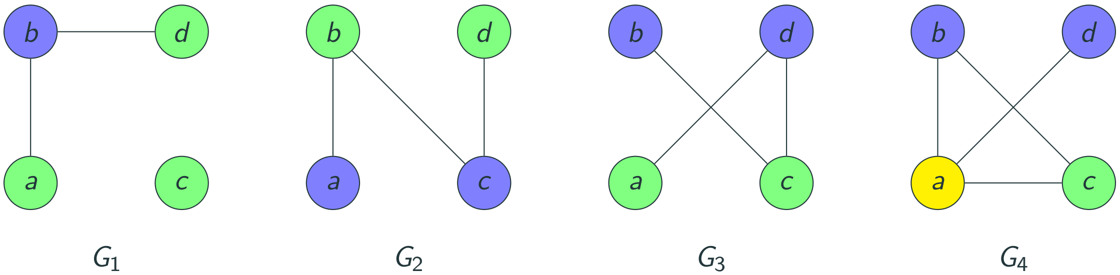
Monday July 7th

Coloring a Temporal Graph



G_1 G_2 G_3 G_4

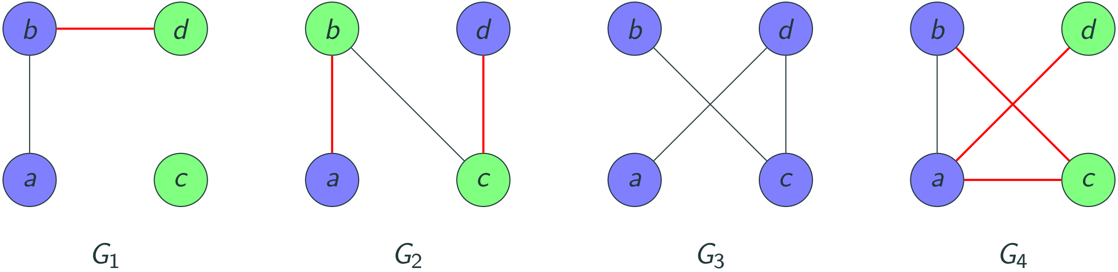
Coloring a Temporal Graph



Color each snapshot ? colors needed : $\max\{\chi(G_i) | i \leq T\}$

3 colors needed

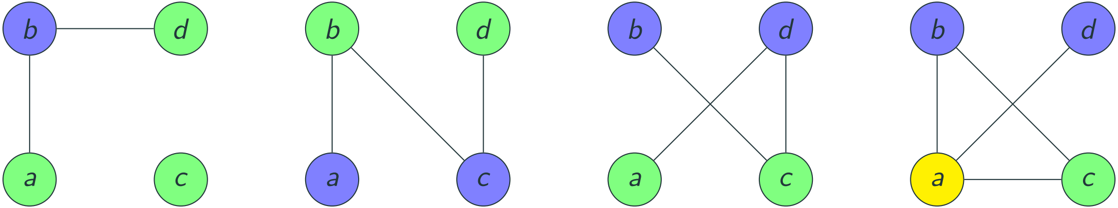
Coloring a Temporal Graph



Mertzios, Molter, Zamaraev -2021 : Each edge must be 2-colored at least once

2 colors are enough

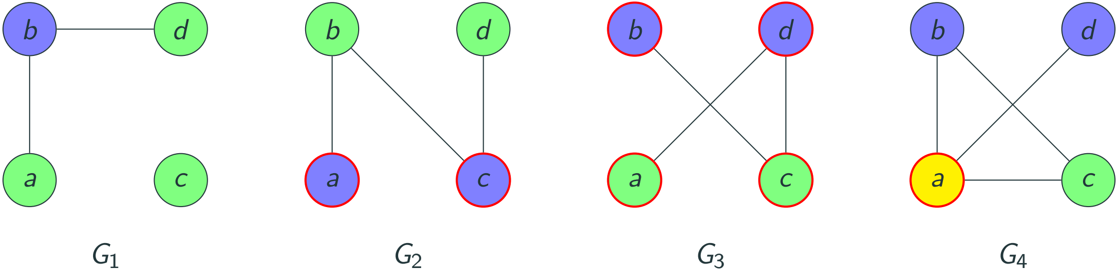
Coloring a Temporal Graph



G_1 G_2 G_3 G_4

Yu, Bar-Noy, Basu, Ramanathan - 2013 : Minimize the number of colors AND color switches

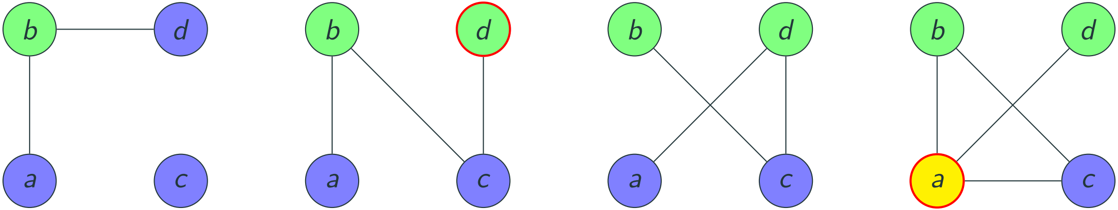
Coloring a Temporal Graph



Yu, Bar-Noy, Basu, Ramanathan - 2013 : Minimize the number of colors AND color switches

3 colors and 7 switches

Coloring a Temporal Graph



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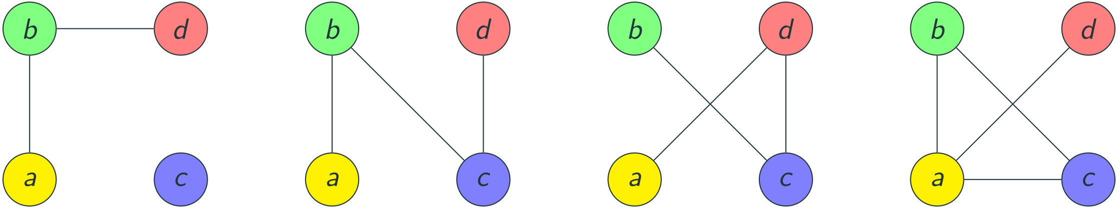
Yu, Bar-Noy, Basu, Ramanathan - 2013 : Minimize the number of colors AND color switches

3 colors and 2 switches

Coloring a Temporal Graph



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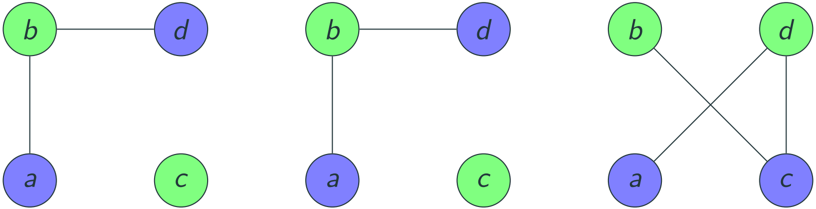
4 colors and 0 switches

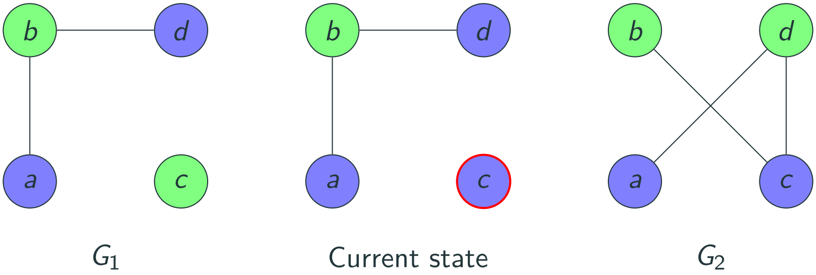
Censors Systems



G_1 Current state G_2

- G_i : graph from censor detectors at time i
- From times i to $i+1$, edges are added, removed, and vertices change colors
- No control on the order in which those operations happen

Censors Systems



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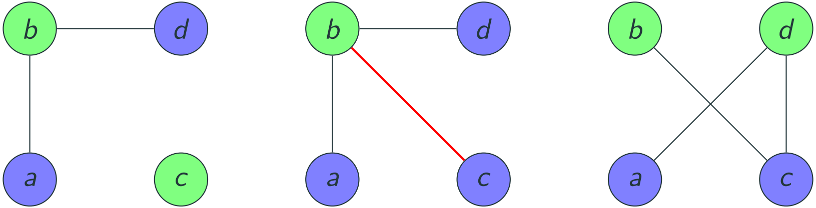

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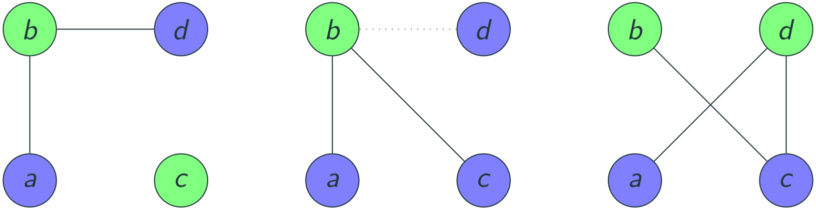

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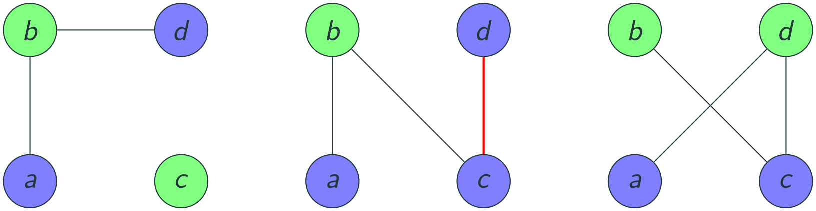

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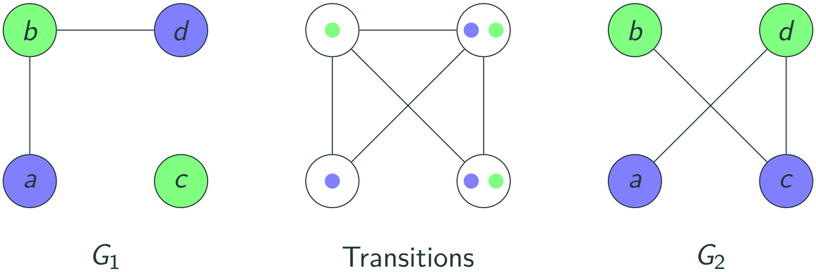

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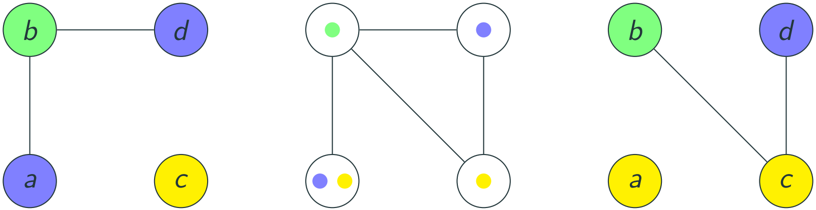
Coloring Compatibility

Two consecutive colorings are **compatible** if :

- c_i and c_{i+1} are proper on $G_i \cup G_{i+1}$
- $\forall uv \in E_i \cup E_{i+1}$, $c_i(u) \neq c_{i+1}(v)$



G_i, c_i $\quad$ $G_i \cup G_{i+1}$, $\{c_i, c_{i+1}\}$ $\quad$ G_{i+1}, c_{i+1}

Temporal Coloring

Temporal chromatic number $\chi^t(\mathcal{G})$ with $\mathcal{G} = (G_1, G_2, \ldots, G_T)$:

- Minimum number of colors for colorings $c_1, c_2, \ldots, c_T$
- $\forall i \leq T$, c_i proper coloring of G_i
- $\forall i < T$, c_i and c_{i+1} compatible on $G_i \cup G_{i+1}$

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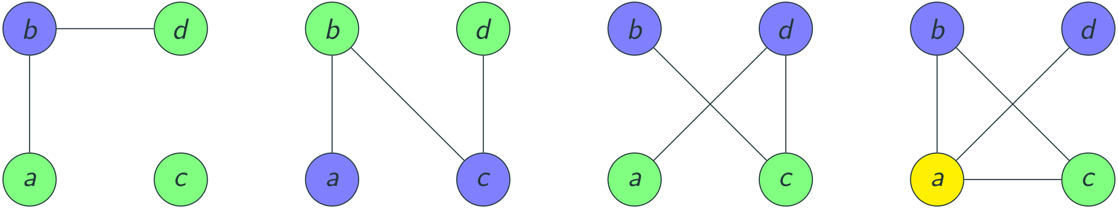
We directly observe :

- $\forall i \in [2, t-1]$, c_i proper coloring of $G_{i-1} \cup G_i \cup G_{i+1}$
- $\max\{\chi(G_{i-1} \cup G_i \cup G_{i+1}) | i \in [2, T-1]\} \leq \chi^t(\mathcal{G}) \leq \chi\left(\bigcup_{i \leq T} G_i\right)$

Upper Bounds - Triplets

From a coloring of each snapshot :

- $\forall i \leq T$, k_i proper coloring of G_i
- K : maximal number of colors used by some k_i
- $\chi^t(\mathcal{G}) \leq K^3$



G_2, k_2 G_3, k_3 G_4, k_4 G_5, k_5

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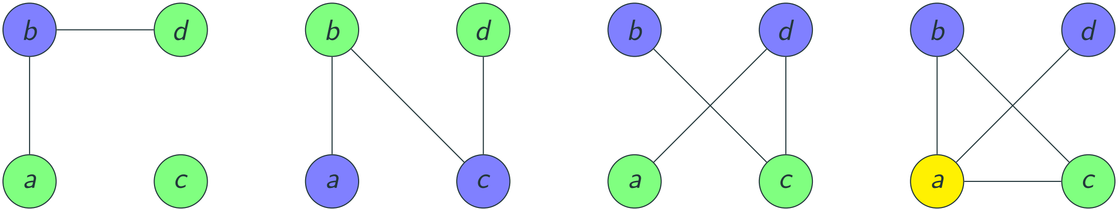
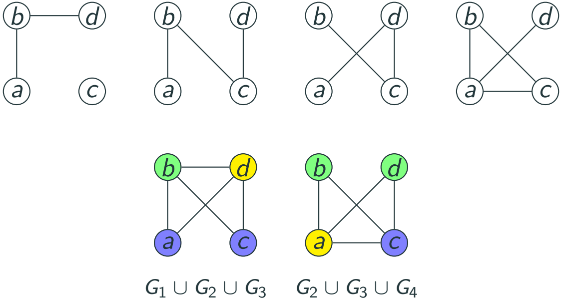


$G_2, c_2 = (k_1, k_2, k_3)$ $G_3, c_3 = (k_4, k_2, k_3)$ $G_4, c_4 = (k_4, k_5, k_3)$ $G_5, c_3 = (k_4, k_5, k_6)$

Upper Bounds - Union of Three

From a coloring of each snapshot :

- $\forall i \in [2, T-1]$, p_i proper coloring of $G_{i-1} \cup G_i \cup G_{i+1}$
- K : maximal number of colors used by some p_i
- $\chi^t(\mathcal{G}) \leq 2K$



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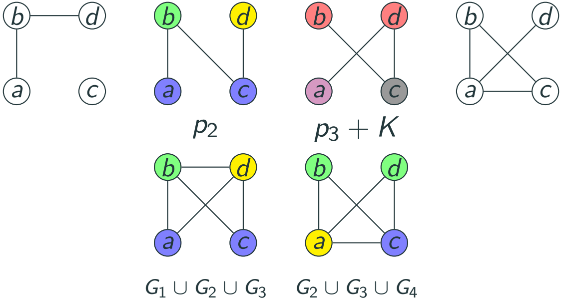
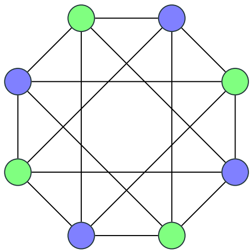


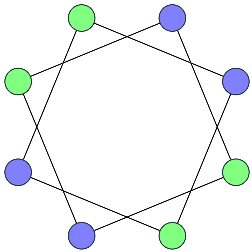
Bipartite Graphs

When each snapshot is a **bipartite graph** :

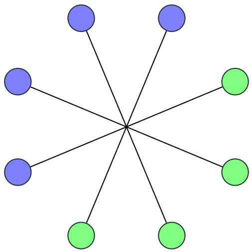
- Triplet upper bound : $k^3 = 8$ colors are enough
- There exists G_1, G_2 and G_3 such as $\chi(G_1 \cup G_2 \cup G_3) = 8$



G_1



G_2

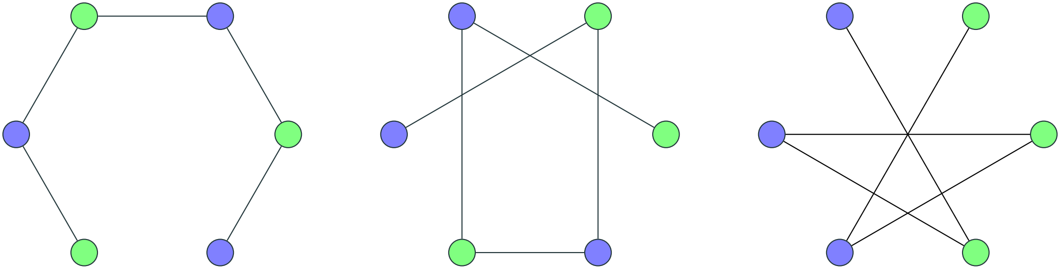


G_3

Trees and Paths

When each snapshot is a **tree** :

- Triplet upper bound : $k^3 = 8$ colors are enough
- Fo all G_1, G_2 and G_3, $\chi(G_1 \cup G_2 \cup G_3) \leq 6$
- There exists G_1, G_2 and G_3 such as $\chi(G_1 \cup G_2 \cup G_3) = 6$

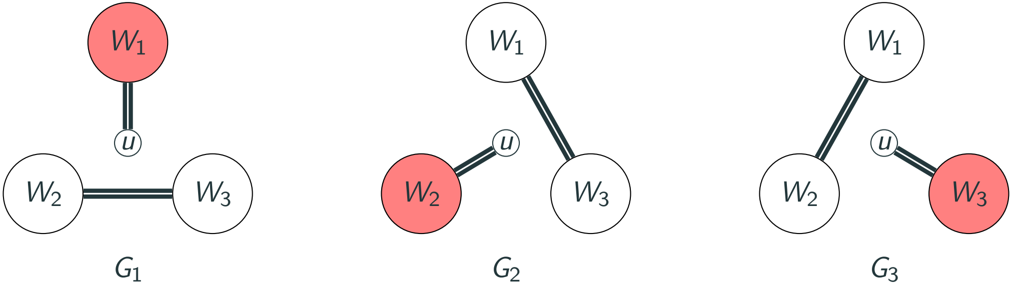


G_1 G_2 G_3

Δ-Bounded Graphs

When each snapshot has **maximal degree Δ** :

- We can always $5\Delta + 1$-color the temporal graph
- Fo all G_1, G_2 and G_3, $\chi(G_1 \cup G_2 \cup G_3) \leq 3\Delta + 1$
- There exists G_1, G_2 and G_3 such as $\chi(G_1 \cup G_2 \cup G_3) = 3\Delta + 1$



d-Degenerate Graphs

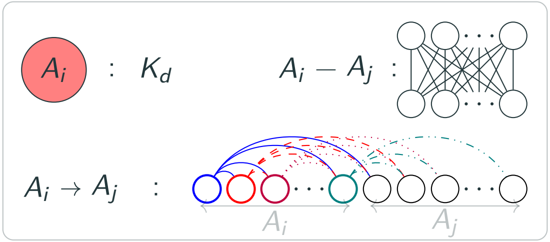
When each snapshot is d-**degenerate** :

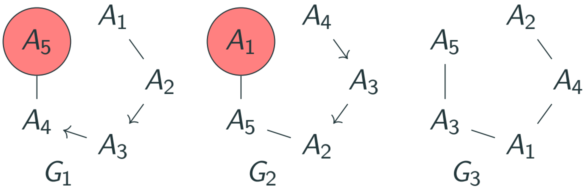
- Triplet and Union of 3 : d^3 and $12d$ colors are enough
- Fo all G_1, G_2 and G_3, $\chi(G_1 \cup G_2 \cup G_3) \leq 6d$
- There exists G_1, G_2 and G_3 such as $\chi(G_1 \cup G_2 \cup G_3) = 5d$

d-Degenerate Graphs

When each snapshot is d-**degenerate** :

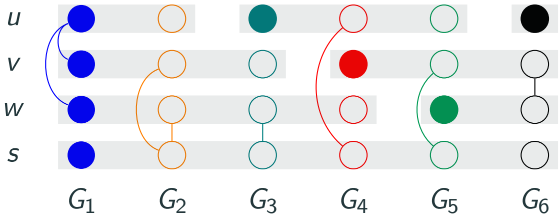
- There exists G_1, G_2 and G_3 such as $\chi(G_1 \cup G_2 \cup G_3) = 5d$



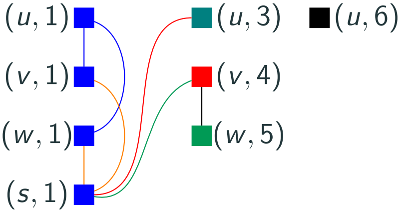


Checking **2-colorability** of a temporal graph is in P :

- To change its color in c_i, a node must be isolated in c_{i-1} and c_i
- Equivalent to 2-color a static graph $col(\mathcal{G})$



$\mathcal{G}$



$col(\mathcal{G})$

Grow Pace

Grow pace of k : $\forall i < T$, $|E_{i+1} \setminus E_i| \leq k$.

For any temporal graph $\mathcal{G}$ with grow pace 1, where each snapshot has bounded degree Δ, we have :

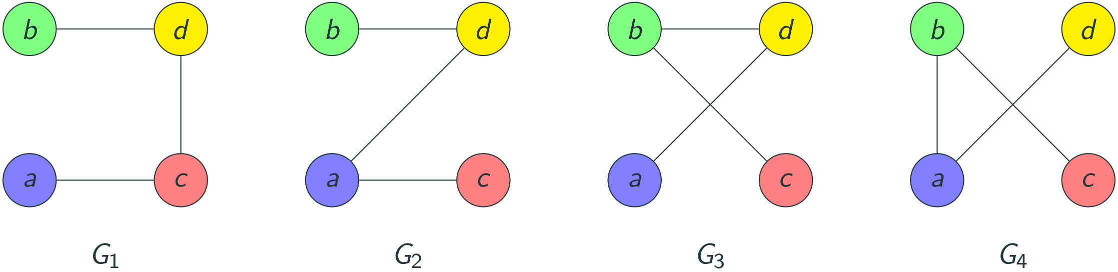
$$\chi^t(\mathcal{G}) \leq \Delta + 2$$

Moreover, for any c_i, coloring of $G_{i-1} \cup G_i \cup G_{i+1}$, we can build c_{i+1} compatible with c_i.

Grow Pace

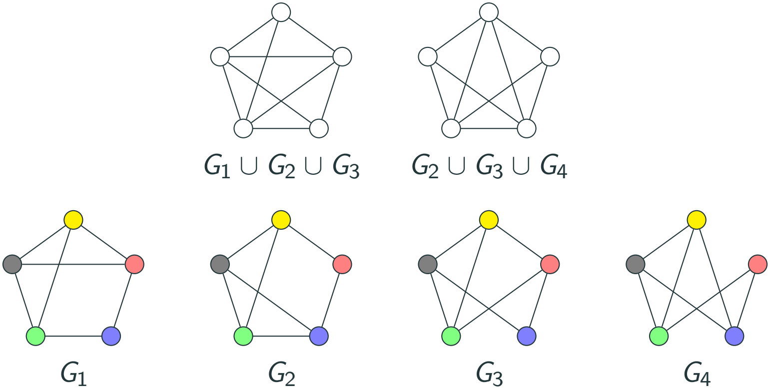
Grow pace of k : $\forall i < T$, $|E_{i+1} \setminus E_i| \leq k$.

- Grow pace 1, bounded degree **2**
- Fo all $\mathcal{G}$, $\chi^t(\mathcal{G}) \leq 4$
- There exists $\mathcal{G} = G_{i \leq 4}$, such as $\chi^t(\mathcal{G}) = 4$



Grow Pace

- Grow pace 1, bounded degree **3**
- Fo all $\mathcal{G}$, $\chi^t(\mathcal{G}) \leq 5$
- There exists $\mathcal{G} = G_{i \leq 4}$, such as $\chi^t(\mathcal{G}) = 5$



Sum Up and Open Questions

- Worst-case scenarios when each snapshot is
  - a tree : $6 \leq \chi^t(\mathcal{G}) \leq 8$
  - of bounded degree Δ : $3\Delta + 1 \leq \chi^t(\mathcal{G}) \leq 5\Delta + 1$
  - d-degenerate : $5d \leq \chi^t(\mathcal{G}) \leq \min(12d, d^3)$
- With grow pace 1 and $\Delta > 5$, are $\Delta + 1$ colors enough ?
- Online vs. centralized temporal coloring : what changes ?
- How to define compatibility for Matchings and Independent Sets ?

https://arxiv.org/abs/2505.10207

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Thank You !

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