Temporal Triadic Closure:

Finding Dense Structures in Social Networks That Evolve

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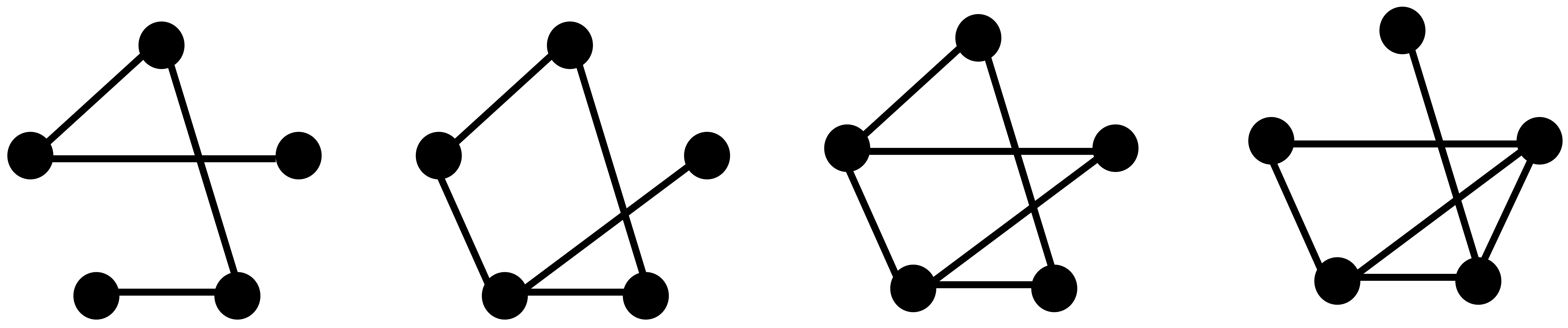
Temporal Graph

A graph in which edges appear and disappear over discrete time-steps



$t = 1$ $\quad$ $t = 2$ $\quad$ $t = 3$ $\quad$ $t = 4$

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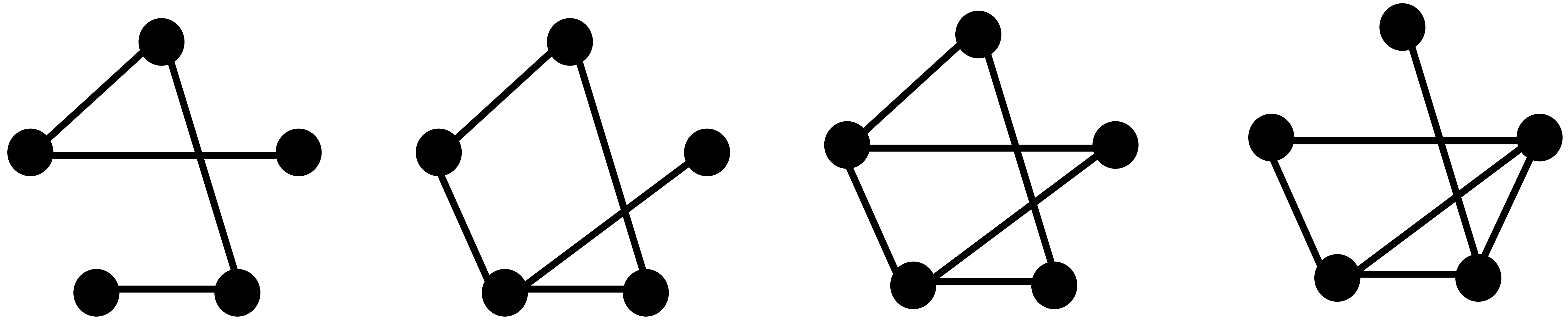

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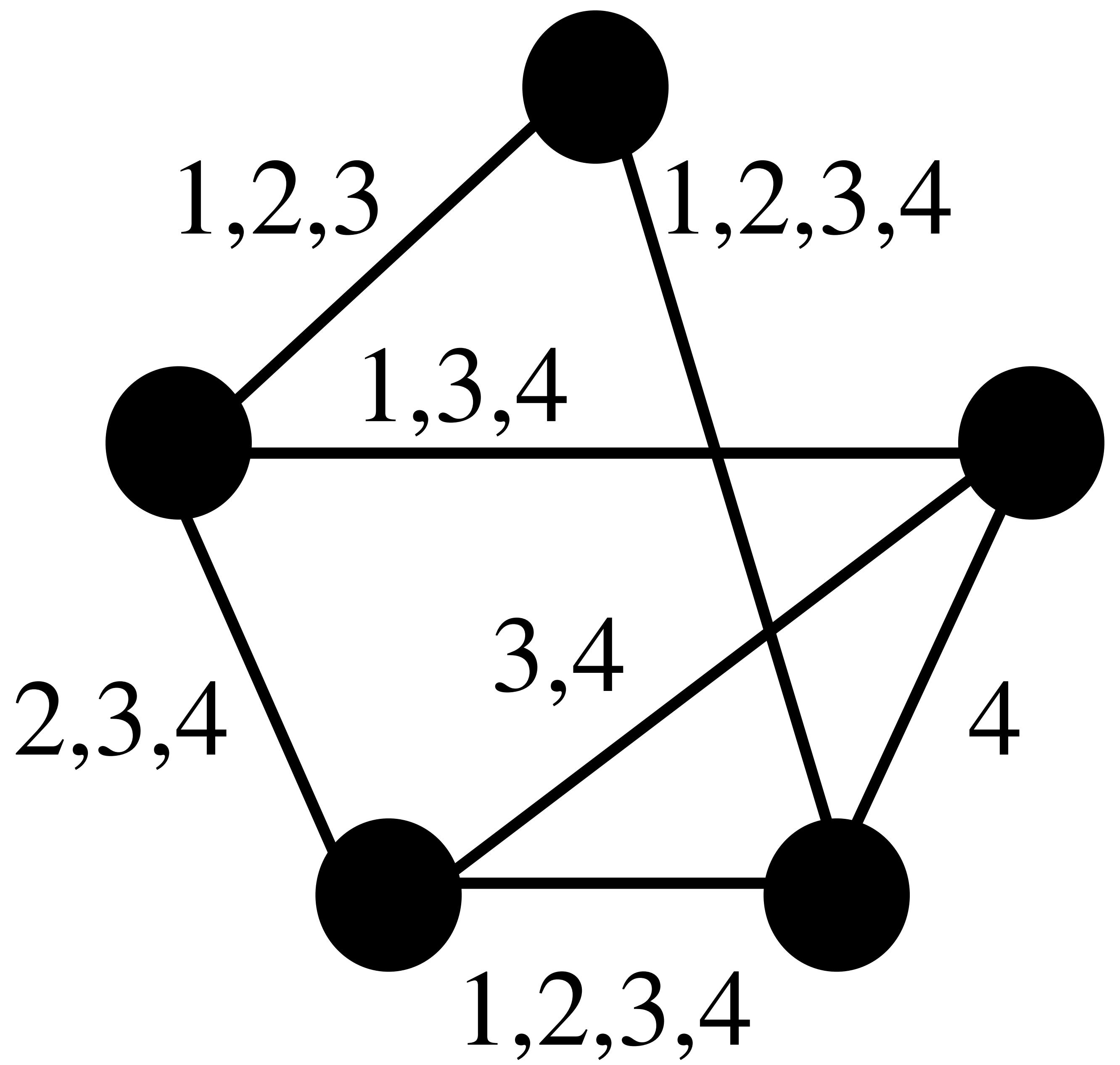

$$\mathscr{G} = (G, \lambda)$$

$\lambda : E(G) \to 2^{\mathbb{N}}$

$\lambda(e) =$ Set of time-steps at which edge e is active

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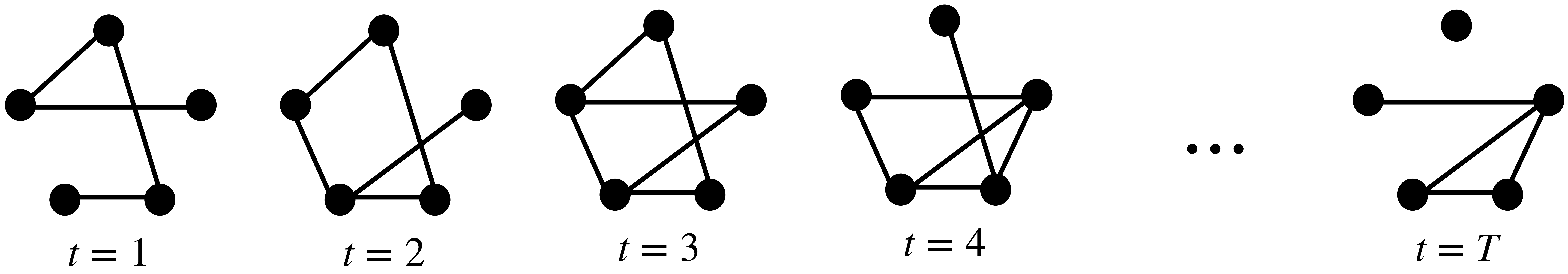
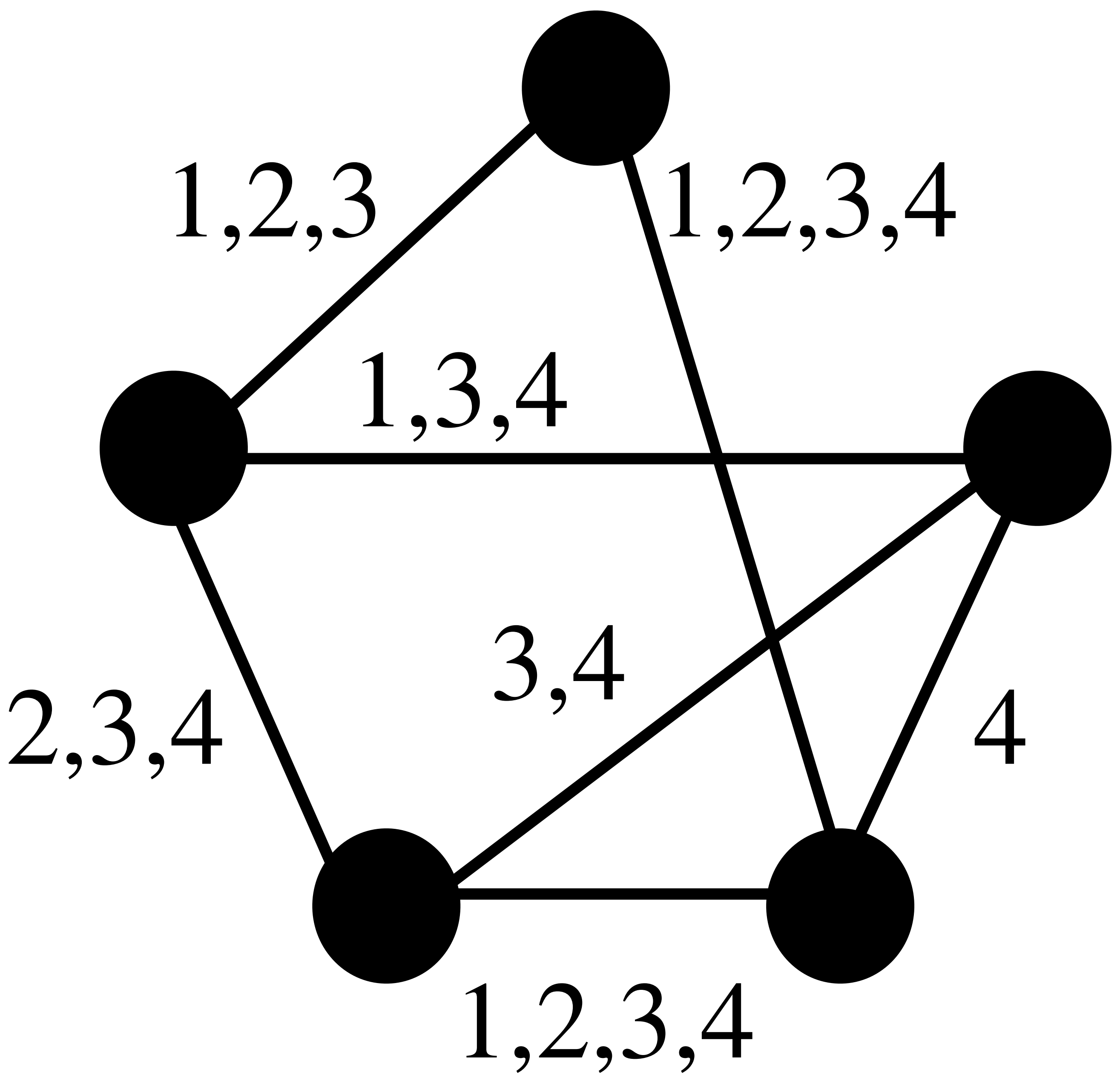


Finite Lifetime



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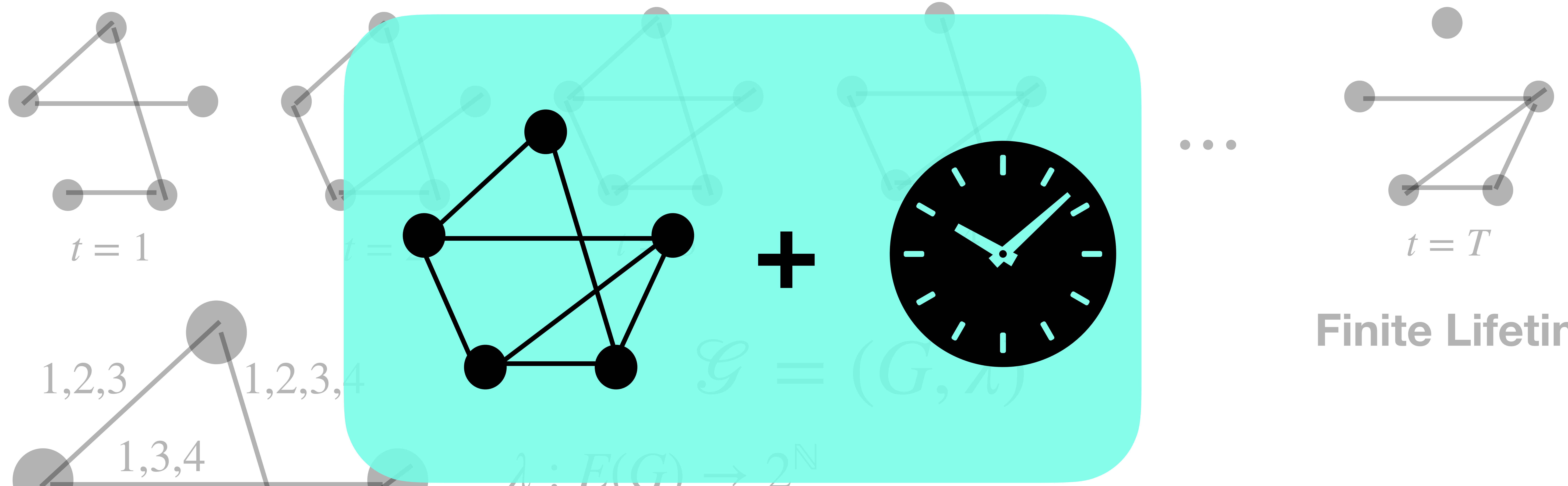

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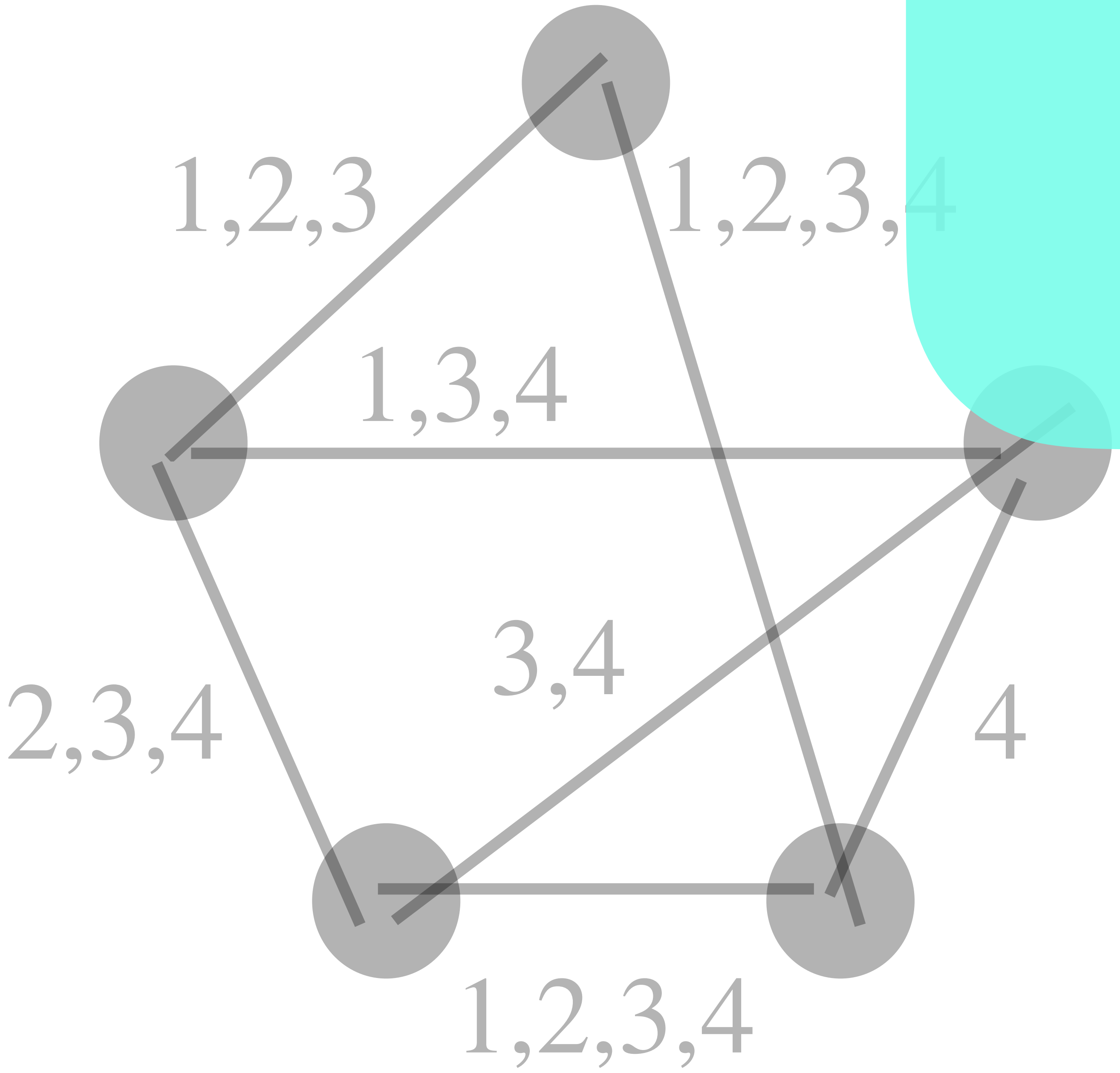

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Translating

Adapting

Generalizing

Extending

Concepts & Algorithms From Graphs to Temporal Graphs

Theorem 1: Let G be a graph. If G satisfies property P, then Q holds.

Can we formulate and prove Theorem 2?

Theorem 2: Let $\mathscr{G}$ be a temporal graph. If $\mathscr{G}$ satisfies a temporal analogue of property P, then a temporal analogue of Q holds.

The hunt for structural parameters for temporal graphs

- NP-hard problems on ***static*** graphs —Structural parameters to the rescue

  NO poly(n) algorithm, but $f(p) \cdot$ **poly**(n) algorithm

  for various parameters p

  Max-Degree
  Degeneracy
  Treewidth
  Cliquewidth
  Forest + p vertices
  …..

- Attempts to translate this algorithmic success to ***temporal*** graphs

  Define structural parameters for temporal graphs

  Treewidth
  Cliquewidth
  Feedback Edge Number
  Nbd Diversity
  ….

The hunt for structural parameters for temporal graphs

- NP-hard problems on ***static*** graphs —Structural parameters to the rescue

A structural parameter inspired by social networks

- Attempts to translate this algorithmic success to ***temporal*** graphs

Define structural parameters for temporal graphs

Treewidth
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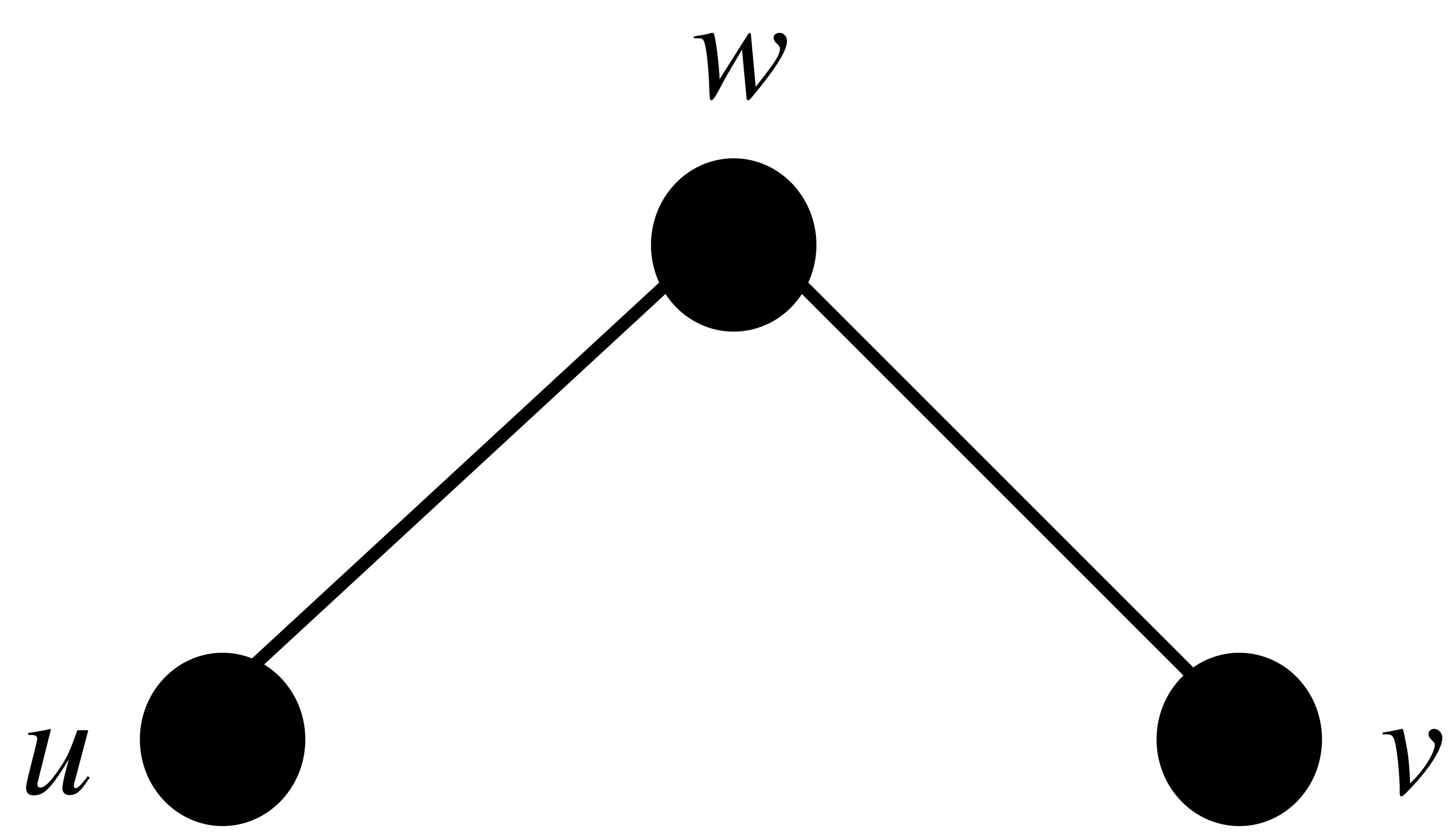
….

Triadic Closure Property

in Social Networks

Friends of friends tend to be friends themselves



A triad

Triadic Closure Property
in Social Networks

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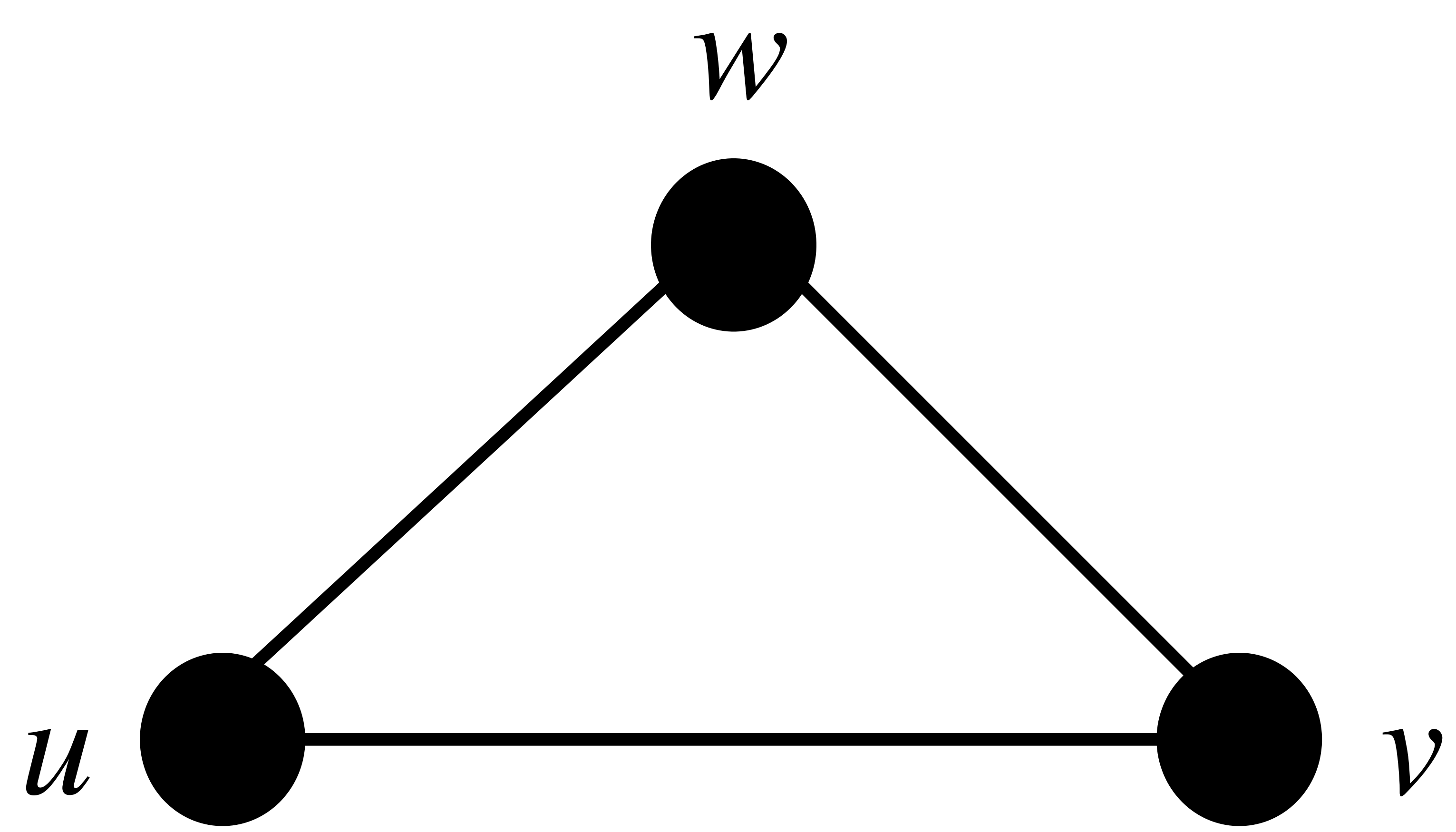
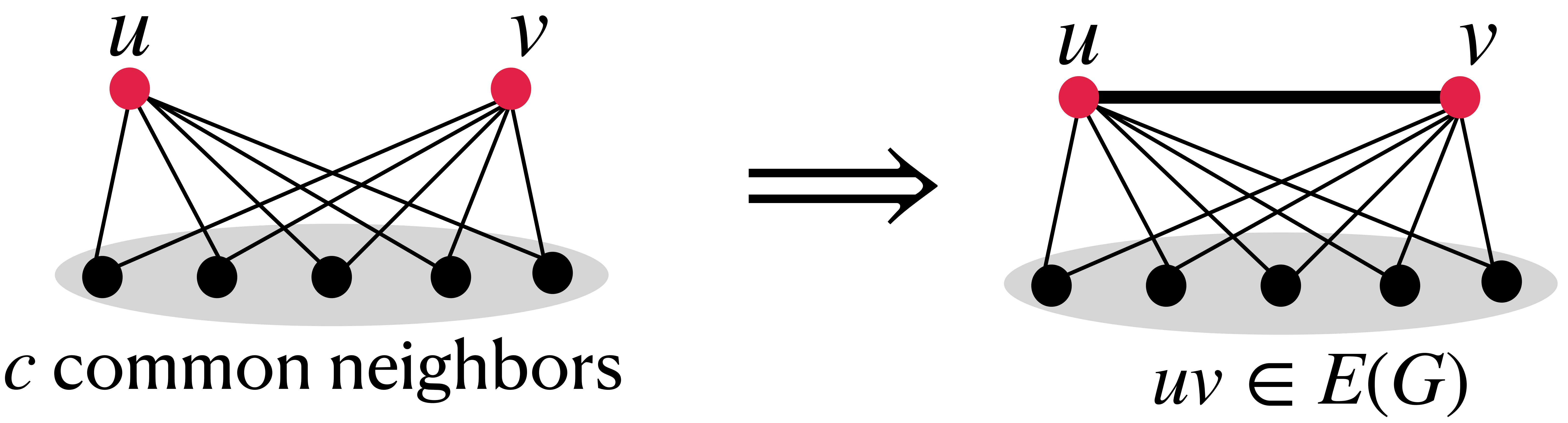


A triad

The triad is closed

c-Closed ~~Temporal~~ Graphs

c-Closed Graphs



$$\text{Closure Number of } G = \min\{c \mid G \text{ is } c\text{-closed}\}$$

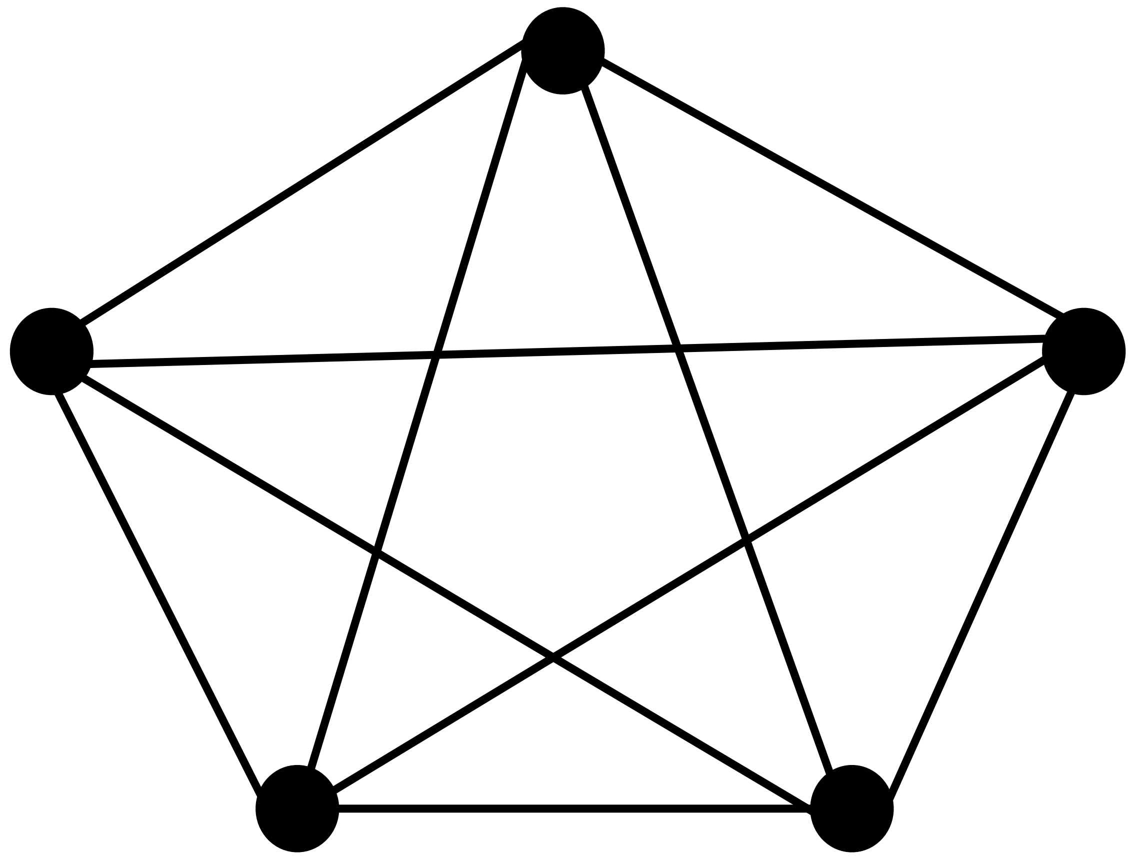
$$= 1 + \max\{0, |N(u) \cap N(v)| \mid uv \notin E(G)\}$$

Closure Number vs. Other Graph Parameters

$$\text{Closure Number of } G = 1 + \max\{0, |N(u) \cap N(v)| \mid uv \notin E(G)\}$$

$$\text{Closure Number of } G \leq 1 + \text{max-degree}(G)$$

K_n



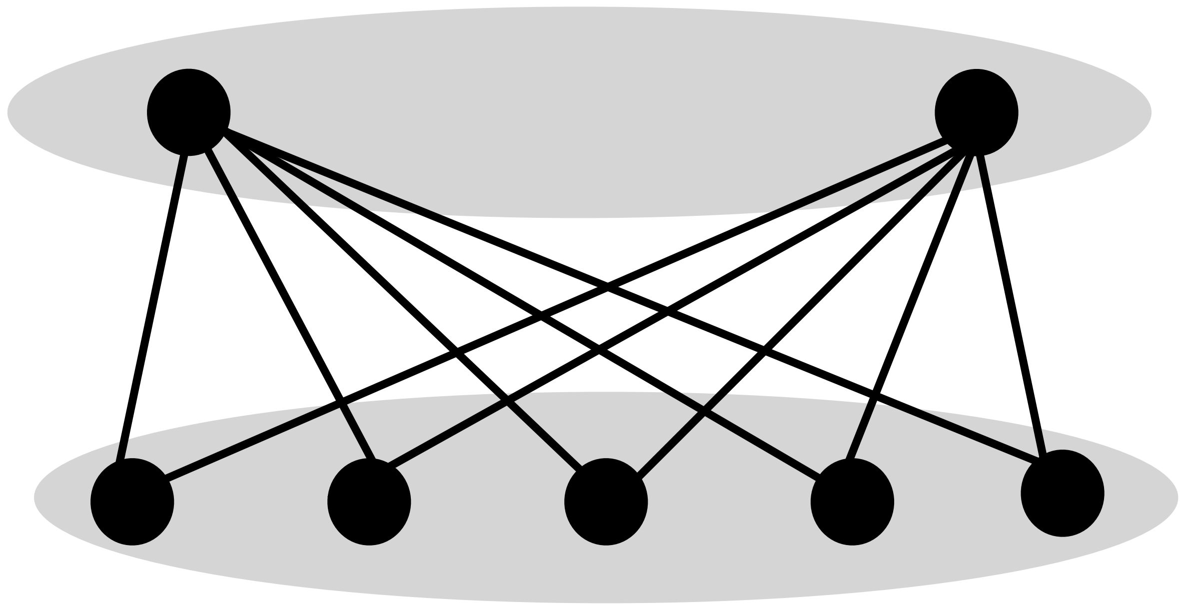
$\mathcal{O}(n^2)$ edges

Treewidth $= n - 1$

Degeneracy $= n - 1$

Max-degree $= n - 1$

Closure Number $= 1$



$K_{2,n-2}$

$\mathcal{O}(n)$ edges

Treewidth $= 2$

Degeneracy $= 2$

Max-degree $= n - 2$

Closure Number $= n - 1$

Static c-closed graphs: A success story

- Introduced by Fox, Roughgarden, Seshadhri, Wei and Wein [ICALP 2018]
- **At most** $3^{c/3} \cdot n^2$ **maximal cliques** (vs. $3^{n/3}$ on general graphs)
- A useful structural parameter — Koana, Komusiewicz, Sommer [ESA 2020]
    - Easy to understand
    - Polynomial-time computable
    - Modest values on real-world networks
    - Captures a property not described by other parameters
    - A number of problems admit efficient algorithms on c-closed graphs
- Inspired a number of papers in the last five years

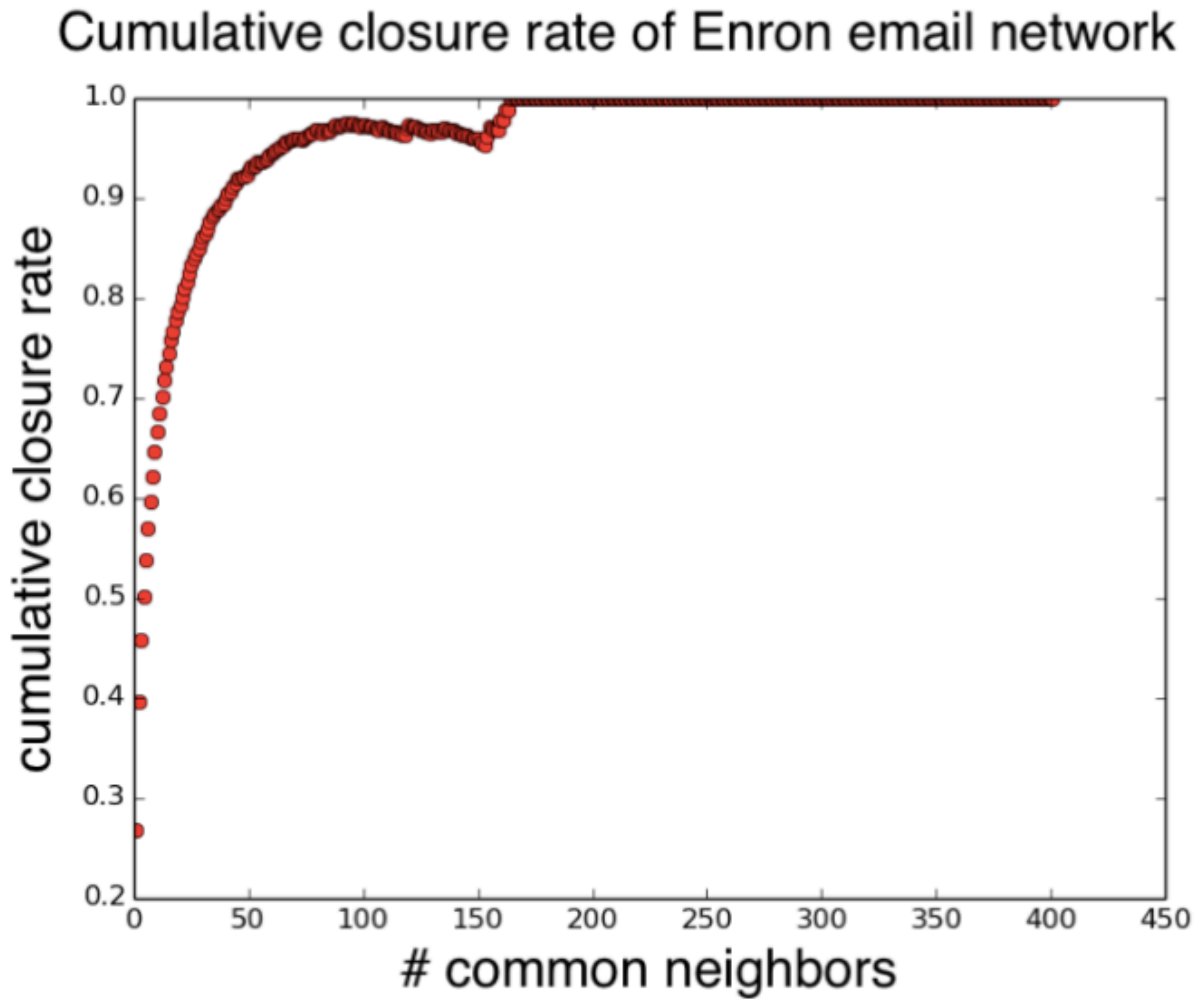
Static c-closed graphs: A success story

	n	m	c	weak c
email-Enron	36692	183831	161	34
p2p-Gnutella04	10876	39994	24	8
wiki-Vote	7115	103689	420	42
ca-GrQc	5242	14496	41	9

[From Fox, Roughgarden, Seshadhri, Wei and Wein (ICALP 2018)]

Static c-closed graphs: A success story

$$y = \frac{\#\{u, v\} \text{ with at least } x \text{ common neighbors and } uv \in E(G)}{\#\{u, v\} \text{ with at least } x \text{ common neighbors}}$$



[Enron email network with 36K nodes and 183K edges: From Fox, Roughgarden, Seshadhri, Wei and Wein (ICALP 2018)]

c-Closed Temporal Graphs

How to define a temporal analogue of c-closed graphs?

Can we prove algorithmic results on temporal c-closed graphs, analogous to the ones on static c-closed graphs?

Recall

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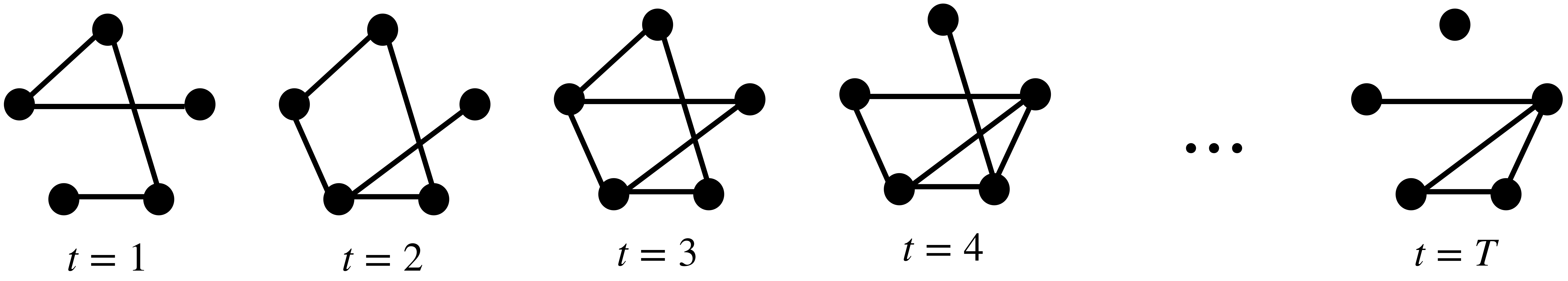

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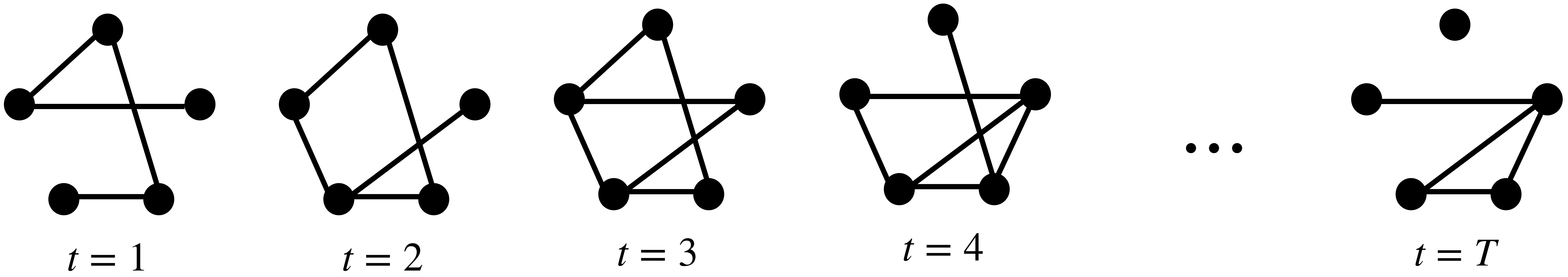

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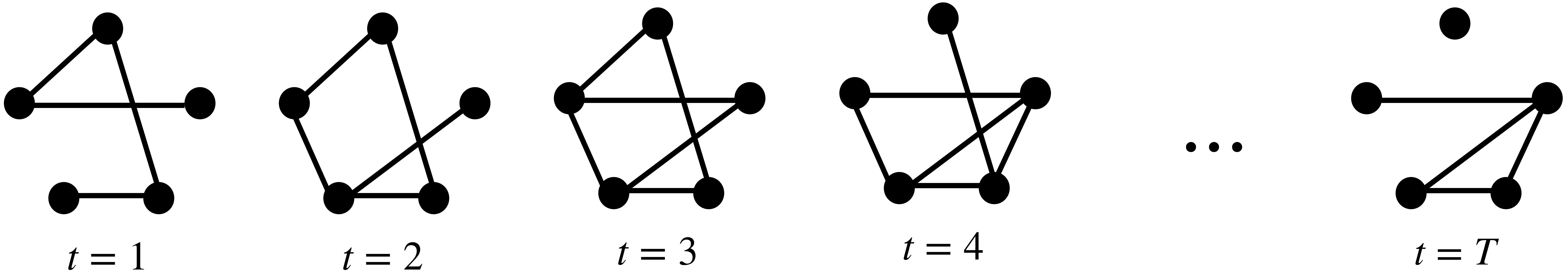
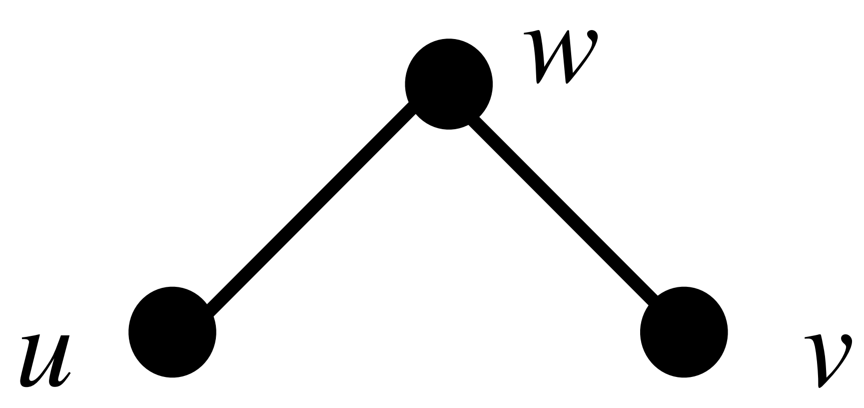


Adjacent during $[a, b]$

uv is active at t for some $t \in [a, b]$

Common neighbor during $[a, b]$



uw is active at t & vw is active at t'

for some $t, t' \in [a, b]$

What we did: c-Closed Temporal Graphs

[Davot, Enright, Madathil, Meeks: AAAI 2025]

- Defined **temporal c-closed** graphs — a formalism of triadic closure property

If two vertices u and v have at least c common neighbors during a short interval of time, then u and v are adjacent to each other around that time.

- Upper bounds for the number of maximal temporal cliques

Every slowly-evolving c-closed temporal graph with n vertices and lifetime T has at most $2^{\mathcal{O}(c)} \cdot n^2 \cdot T$ maximal temporal cliques.

$2^{\Omega(n)} \cdot T$

- Enumerate all maximal temporal cliques in time $2^{\mathcal{O}(c)} \cdot n^{\mathcal{O}(1)} \cdot T$.

- More general results — other dense subgraphs, weakly c-closed temporal graphs.

- Empirical analysis of a handful of small real-world networks: c-closed for modest values of c

Quick Overview of Technical Details

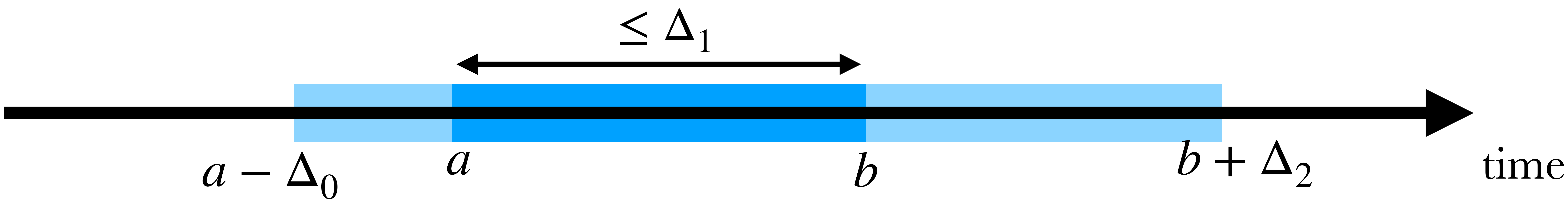
c-Closed Temporal Graphs

If two vertices u and v have at least c common neighbors during a short interval of time,
then u and v are adjacent to each other around that time.

(Almost) Formal Definition of $(\Delta_0, \Delta_1, \Delta_2, c)$-closed temporal graphs

For every two distinct vertices u and v,
and every time-interval $[a, b]$ with $b - a \leq \Delta_1$,

if u and v have at least c common neighbors during $[a, b]$,
then u and v are adjacent to each other during $[a - \Delta_0, b + \Delta_2]$.



Temporal Cliques: Δ-Cliques

[Viard, Latapy, Magnien: Computing maximal cliques in link streams. TCS, 2016]

Clique in a static graph: $X \subseteq V(G)$ such that $xy \in E(G)$ for every $x, y \in X$

Clique in a temporal graph: $(X, [p, q])$

$X \subseteq V(G)$ and $[p, q]$ is a time-interval

such that for every $x, y \in X$ and

for every time-interval $[t, t + \Delta] \subseteq [p, q]$,

The edge xy is active during $[t, t + \Delta]$

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#Maximal Δ-cliques could be as large as $2^n \cdot T$

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Can we bound #Maximal Δ-cliques by $f(c) \cdot \mathbf{poly}(n) \cdot T$?

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#Maximal Δ-cliques could be as large as $2^n \cdot T$

Can we bound #Maximal Δ-cliques by $f(c) \cdot$ poly$(n) \cdot T$?

Yes, if the graph evolves slowly

No, otherwise

A (Fast-Evolving) Temporal c-Closed Graph with $\Omega(2^n)$ maximal Δ-cliques

Construct an n-vertex temporal graph as follows.

$X_1, X_2, \ldots, X_{2^n-(n+1)}$: the $2^n - (n+1)$ subsets of vertices of size at least 2

Choose time-steps $t_1 < t_2 < \cdots < t_{2^n-(n+1)}$ such that $t_{i+1} - t_i > 3\Delta$.

At time-step t_i, the set X_i induces a clique.

1-closed

Then $(X_i, [t_i - \Delta, t_i + \Delta])$ is a maximal Δ-clique.

Slow Evolution: Small instability

Between consecutive time-steps, the neighborhood of each vertex changes very little

η-unstable temporal graph

$$|N_t(v) \setminus N_{t+1}(v)| \leq \eta$$

And

$$|N_{t+1}(v) \setminus N_t(v)| \leq \eta$$

Our Bound for #Maximal Δ-Cliques

$$2^{c+2\eta\Delta} \cdot n^2 \cdot T$$

Outline of the proof:

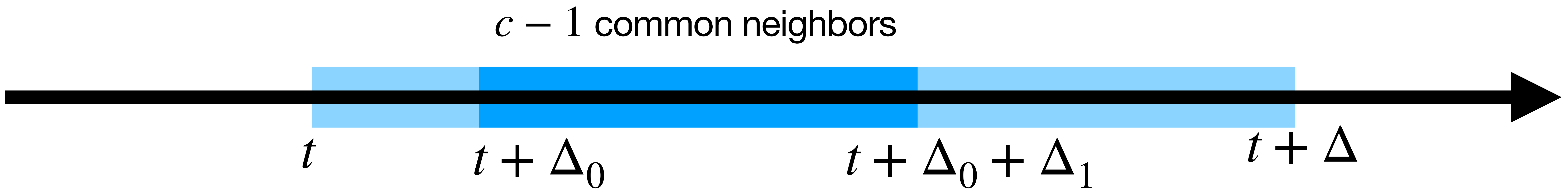
Associate every maximal Δ-clique with

a pair of vertices u, v,

a time-step t

a subset of the common neighborhood of u and v during $[t, t + \Delta]$

such that uv is not active during $[t, t + \Delta]$



$c - 1 + 2\eta\Delta$ common neighbors

Empirical Results

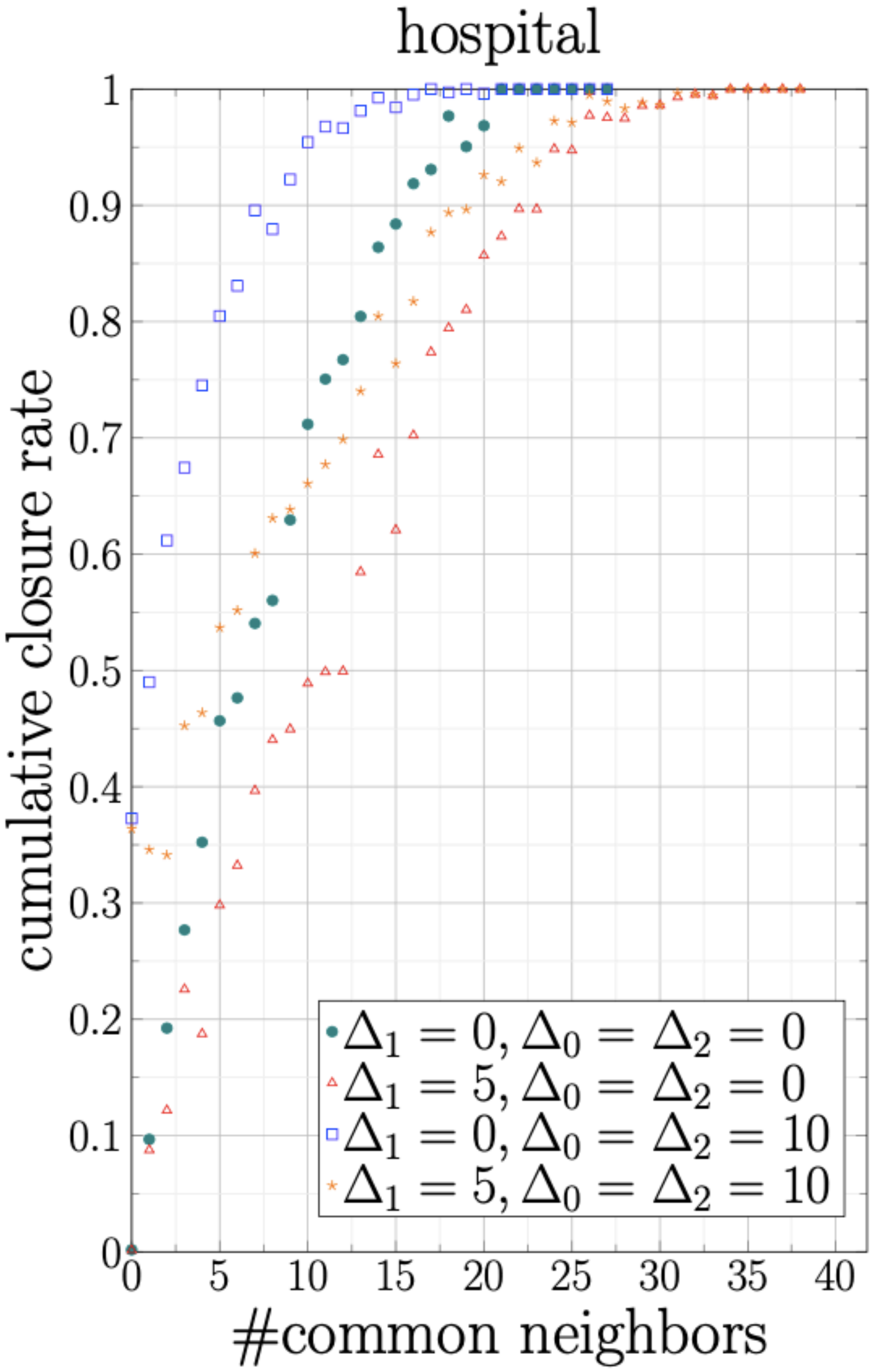
sociopatterns.org

- Seven contact networks
    - #vertices in the range 21–217
    - #edges in the range 54–4274
    - #time-steps in the range 27—275
    - c values in the range 8—30

Temporal c value is smaller than the static c value

$$y = \frac{\#(\{u,v\},a),\ \text{where } u \text{ and } v \text{ have at least } x \text{ common neighbors during } [a, a+\Delta_1] \text{ and } uv \text{ is active during } [a-\Delta_0, a+\Delta_1+\Delta_2])}{\#(\{u,v\},a),\ \text{where } u \text{ and } v \text{ have at least } x \text{ common neighbors during } [a, a+\Delta_1]}$$

#vertices = 73,
#edges = 1381,
lifetime =71



Summary and Next Steps

- Defined c-closed (and weakly c-closed) temporal graphs
- Bounded the number of maximal cliques
- Similar bounds for k-plexes and k-defective cliques
- Introduced notions of stability

- Bounds for other dense subgraphs — (dense := complement of sparse)?
- Weaker notions of stability?
- Usefulness of c-closure in designing algorithms for temporal graph problems?
- Detailed empirical study?

Thank You