Paths and cliques in random temporal graphs

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based on joint work with
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temporal graphs

A temporal graph $G = (V, E, \pi)$ is a finite simple graph together with an ordering $\pi : E \to \{1, 2, \ldots, |E|\}$ of the edges.

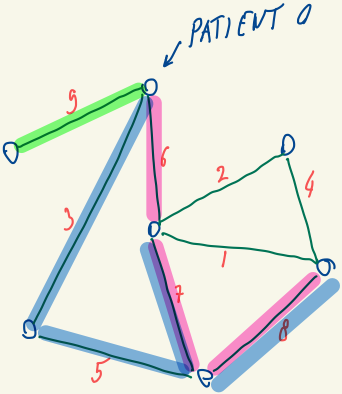
The edges have time stamps.

Edge e precedes edge f if $\pi(e) < \pi(f)$.

Temporal graphs model time-dependent propagation processes such as infection processes.

An infection spreads along monotone increasing paths.

temporal graphs



random simple temporal graphs

Today $G \sim \mathcal{G}(n, p)$ is an Erdős–Rényi random graph.

Such a random graph may be generated by assigning i.i.d. random labels to each edge of the complete graph K_n.

The label of edge $\{i, j\}$ is a uniform $[0, 1]$ random variable $U_{i,j}$.

An edge is kept if and only if $U_{i,j} \leq p$.

Denote the random temporal graph by $G(U)$.

connectivity

This model was formally introduced by Casteigts, Raskin, Renken, and Zamaraev (2022) and Becker, Casteigts, Crescenzi, Kodrić, Renken, Raskin, and Zamaraev (2022).
They prove that, with high probability,

- a typical pair of vertices is connected by an increasing path if $p \geq (1+\epsilon)\log n/n$ and disconnected if $p \leq (1-\epsilon)\log n/n$.
- a typical vertex can reach all other vertices if $p \geq (2+\epsilon)\log n/n$ and cannot reach all of them if $p \leq (2-\epsilon)\log n/n$.
- any pair of vertices are connected if $p \geq (3+\epsilon)\log n/n$ but not all of them are connected if $p \leq (3-\epsilon)\log n/n$.

longest and shortest monotone paths

Define $\ell(i, j)$ and $L(i, j)$ as the minimum and maximum length of any increasing path from i to j.

We study

- $L(1, 2)$ and $\ell(1, 2)$; the lengths of the longest and shortest increasing paths between two fixed vertices.
- $\max_{j \in \{2,\ldots,n\}} L(1, j)$ and $\max_{j \in \{2,\ldots,n\}} \ell(1, j)$; the maximum length of the longest and shortest increasing paths starting at a fixed vertex;
- $\max_{i,j \in [n]} L(i, j)$ and $\max_{i,j \in [n]} \ell(i, j)$; the maximal length of the longest and shortest increasing paths.

longest monotone paths

Angel, Ferber, Sudakov, and Tassion (2020) showed that if $p = o(n)$ and, $pn/\log n \to \infty$ then, with high probability,

$$\max_{i,j \in [n]} L(i,j) \sim enp \ .$$

This excludes the "interesting" regime $p \sim c \log n/n$.

In fact, when $pn/\log n \to \infty$,

$$L(1,2) \sim enp \ .$$

proof of $L(1,2) \sim enp$

Partition $[0,p)$ as $[0, 2\log n/n) \cup [2\log n/n, p - 2\log n/n) \cup [p - 2\log n/n, p)$ and let U_1, U_2, U_3 be the collections of edge weights falling in the corresponding intervals.

This decomposes $G(U)$ into the union of three graphs $G(U_1), G(U_2), G(U_3)$.

The longest monotone path in $G(U_2)$ has length $\sim enp$, say from i^* to j^*.

But in $G(U_1)$ there is a path from vertex 1 to i^* and in $G(U_3)$ there is a path from vertex j^* to 2.

some key constants

For any $c > 0$, define

$$\alpha(c) = \inf\{x > 0 : x \log(x/ec) = 1\}$$

and for $c > 1$,

$$\beta(c) = \sup\{x > 0 : x \log(x/ec) = -1\}$$

$$\gamma(c) = \inf\{x > 0 : x \log(x/ec) = -1\}$$

some key constants

The equation $x\log(x/ec) = -1$ has at most two solutions for $c > 0$, these are $\beta(c)$ and $\gamma(c)$.

When $c = 1$, there is only one solution and $\beta(1) = \gamma(1) = 1$.

For $c < 1$ there is no solution, for $c > 1$ there are two.

The equation $x\log(x/ec) = 1$ has a unique solution for all $c > 0$.

Note that as $c \to \infty$, $\alpha(c)/c \to e$, $\beta(c)/c \to e$, and $\gamma(c)/c \to 0$.

An example value is $\alpha(1) \approx 3.5911$.

longest monotone paths

Suppose $p = c \log n/n$. With high probability,

- if $c \in (0,1)$, there is no increasing path between 1 and 2, and if $c \geq 1$, $L(1,2) \sim \beta(c) \log n$;
- for all $c > 0$, $\max_{j \in \{2,\dots,n\}} L(1,j) \sim ec \log n$;
- for all $c > 0$, $\max_{i,j \in [n]} L(i,j) \sim \alpha(c) \log n$.

shortest monotone paths

Let $p = c \log n / n$. With high probability,

- for $c > 1$, $\ell(1, 2) \sim \gamma(c) \log n$;
- for $c > 2$, $\max_{i \in [n]} \ell(1, i) \sim \gamma(c - 1) \log n$;
- for $c > 3$, $\max_{i,j \in [n]} \ell(i, j) \sim \gamma(c - 2) \log n$.

upper bounds: first moment considerations

Let X_k be the number of increasing paths of length k. Then

$$\mathbb{E}X_k = \binom{n}{k+1}(k+1)!\frac{p^k}{k!} \sim n\left(\frac{nep}{k}\right)^k$$

Similarly, for the number Y_k of increasing paths of length k starting at vertex 1 and for the number Z_k of increasing paths of length k vertex 1 to vertex 2,

$$\mathbb{E}Y_k = \binom{n-1}{k}k!\frac{p^k}{k!} \sim \left(\frac{nep}{k}\right)^k$$

$$\mathbb{E}Z_k = \binom{n-2}{k-1}(k-1)!\frac{p^k}{k!} \sim \frac{1}{n}\left(\frac{nep}{k}\right)^k$$

The upper bounds follow simply from these identities.

We prove the lower bounds of the three statements by three different techniques.

proof of $\max_j L(1,j) \geq ec(1-o(1))\log n$

Since the bound is linear in c, it suffices to prove the lower bound for $c \leq 1$. Otherwise we may decompose the graph into $\lceil c \rceil$ disjoint layers and concatenate the paths.

When $c \leq 1$, one can show that the graph contains a uniform random recursive tree of size $n^{c(1-o(1))}$, rooted at vertex 1 such that all paths of the tree starting at vertex 1 are monotone.

The construction of the tree is similar to the shortest-path tree. We need to discard a small number of vertices in order to keep monotonicity.

Since the height of the URRT is $\sim e \log n^{c(1-o(1))}$, we have a monotone path of desired length.

This also shows that at least $n^{c(1-o(1))}$ can be reached from vertex 1.

proof of $\max_{i,j} L(i,j) \geq \alpha(c)(1 - o(1)) \log n$

Since the expected number of monotone paths of length $\alpha(c)(1 - o(1)) \log n$ goes to infinity, it is natural to resort to the second moment method.

However, the second moment is too large due to the many ways paths can intersect.

We borrow ideas from Addario-Berry, Broutin, and Lugosi (2010) and apply the second moment method to a restricted class of paths.

building short and long monotone paths

Suppose $c > 1$. It still remains to show that

$$\ell(1,2) \leq (\gamma(c) + o(1)) \log n$$

and

$$L(1,2) \geq (\beta(c) - o(1)) \log n$$

Recall that $\gamma(c) < \beta(c)$ are the two solutions of
$x \log(x/ec) = -1.$
We prove that for any $x \in (\gamma(c), \beta(c))$, whp there exists an increasing path between 1 and 2 containing $\sim x \log n$ edges.

Note that $\gamma(1) = \beta(1) = 1$ so for $c \approx 1$, all monotone paths between 1 and 2 have about the same length $\log n$.

building short and long monotone paths

We look for increasing paths from vertex $\mathbf{1}$ such that labels increase as they should to have length $x \log n$.

Similarly, we look for decreasing paths from vertex $\mathbf{2}$

We conduct this search up to distance $\frac{1}{2} x \log n$.

We show that the two sets of end points of the path must intersect, because the sets at distance $\frac{x}{2} \log n$ are of size at least $n^{1/2}$.

the temporal diameter

The last argument is to show why

- for $c > 2$, $\max_{i\in[n]} \ell(1,i) \sim \gamma(c-1)\log n$;
- for $c > 3$, $\max_{i,j\in[n]} \ell(i,j) \sim \gamma(c-2)\log n$.

For $c > 2$, if we partition $(0, c\log n/n) = I_1 \cup I_2$ with I_2 of length $(1-\epsilon)\log n/n$, then $G(I_2)$ has an isolated vertex and in $G(I_1)$ shortest paths are of length at least $\gamma(c-1)\log n$.

temporal cliques

A set of vertices forms a temporal clique if there is an increasing path from any vertex to any other vertex.

We are interested in the size of the largest temporal clique.

Becker, Casteigts, Crescenzi, Kodrić, Renken, Raskin, and Zamaraev (2022) show that:

if $p > (1+\epsilon)\log(n)/n$, then there is a temporal clique of size $n - o(n)$, with high probability;

if $p < (1-\epsilon)\log(n)/n$, then the largest temporal clique is of size $o(n)$, with high probability.

temporal cliques in the sub-critical regime

Let $p = c\log n/n$ with $c < 1$. With high probability, the largest temporal clique is of size at most

$$\left\lceil \frac{1}{1-c} + 1 \right\rceil$$

Thus, the temporal clique number jumps from constant to $n - o(n)$.

For $c \leq 1/2$, the upper bound is 3 which is optimal, since in this range, $G(n,p)$ has (many) trianges.

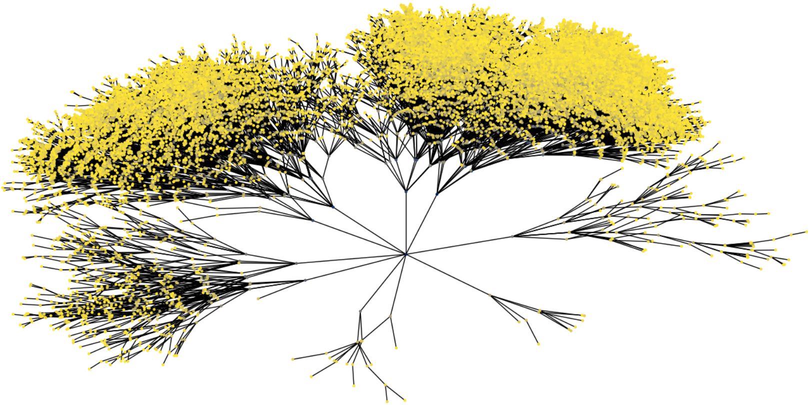
uniform temporal trees

Assigning independent uniform $[0,1]$ labels to the edges of a rooted complete infinite n-ary tree.

Keep only those vertices for which the path from the root to the vertex has decreasing edge labels.

In the p-percolated temporal tree $\mathcal{T}_{n,p}$, only edges with labels below p are kept.

uniform temporal trees



uniform temporal trees–the size

The expected size is easily to calculate: each one of the n^k vertices at depth k are kept with probability $p^k/k!$, so

$$\mathbb{E}|\mathcal{T}_{n,p}| = \sum_{k=0}^{\infty} \frac{(np)^k}{k!} = e^{np}$$

We also have

$$\frac{|\mathcal{T}_{n,p}|}{e^{np}} \xrightarrow{\mathcal{L}} E \quad \text{as } n \to \infty ,$$

where E is an exponential(1) random variable.

uniform temporal trees–distribution of mass at the root

Let v_i be the child of the root with the i-th largest label. Then for any $m \geq 1$,

$$\left(\frac{|\mathcal{T}_{n,p}(v_1)|}{e^{np}}, \ldots, \frac{|\mathcal{T}_{n,p}(v_m)|}{e^{np}}\right) \tag{1}$$

$$\xrightarrow{\mathcal{L}} (E_1 U_1, E_2 U_1 U_2, \ldots, E_m U_1 \cdots U_m) \quad \text{as } n \to \infty \ , \tag{2}$$

where E_k are i.i.d. exponential (1) and U_k are i.i.d. uniform $[0, 1]$.

uniform temporal trees–height and depth

Let $H_{n,p}$ denote the height of a percolated uniform temporal tree $\mathcal{T}_{n,p}$. Then

$$\frac{H_{n,p}}{np} \xrightarrow{\mathbb{P}} e \quad \text{as } n \to \infty.$$

Let let $D_{n,p}$ denote the depth of a uniformly chosen vertex in a percolated uniform temporal tree $\mathcal{T}_{n,p}$. Then

$$\frac{D_{n,p}}{np} \xrightarrow{\mathbb{P}} 1 \quad \text{as } n \to \infty.$$

uniform temporal trees–degree distribution

For $k \geq 0$, let $L_{n,k}$ denote the number of vertices of out-degree k in a percolated uniform temporal tree $\mathcal{T}_{n,p}$. Then

$$\frac{\mathbb{E}L_{n,k}}{e^{np}} \to 2^{-(k+1)} \text{ as } n \to \infty.$$

Height, depth, and degree distribution as in a uniform random recursive tree of size e^{np}.

some further questions

- Length of longest path when p is constant?
- What happens in the critical window?
- Optimality of the clique number bound for $c \in (1/2, 1)$?
- Different random graph models.
- Models with recovery, reinfection.
- Super spreader events.
- Statistical questions.

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