Temporal Exploration of Random Spanning Tree Models

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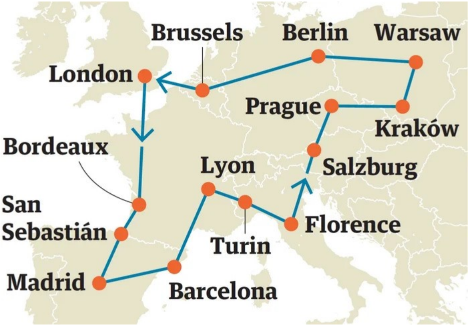
Exploration

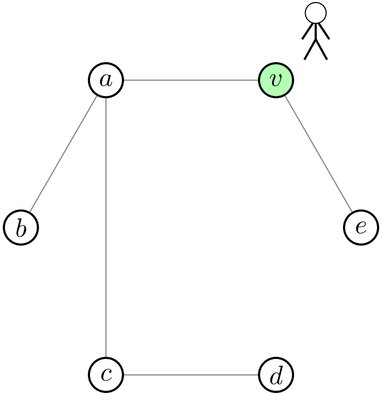
Task: visit all European cities by train before your interrail pass expires...

interrail
eurail



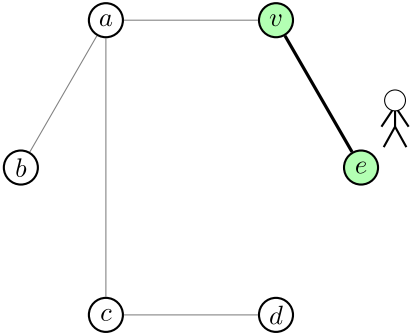
The problem of minimising the total time is called the *Graphical Travelling Salesperson Problem.*

Exploration



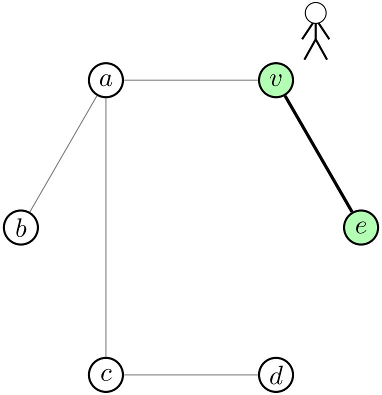
$t = 0$

Exploration



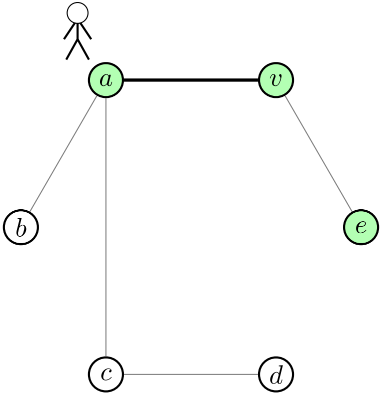
$t = 1$

Exploration



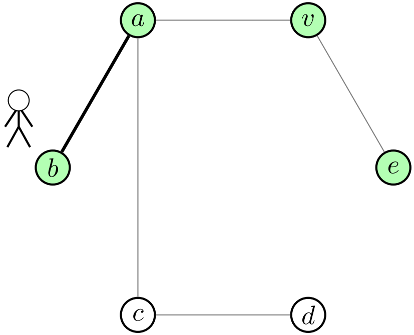
$t = 2$

Exploration



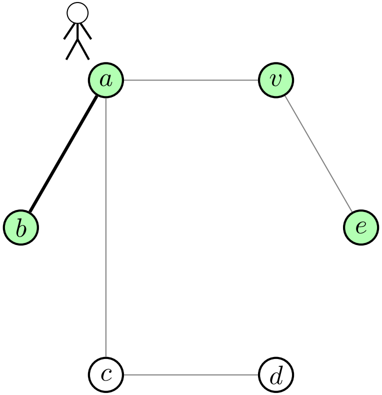
$t = 3$

Exploration



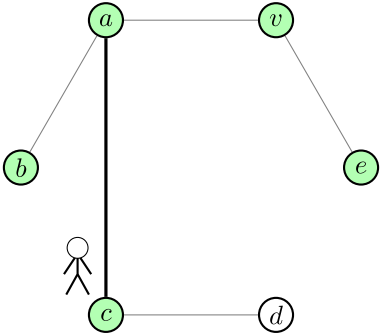
$t = 4$

Exploration



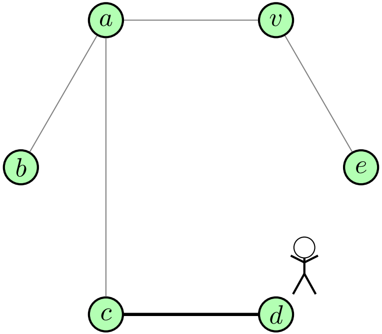
$t = 5$

Exploration



$t = 6$

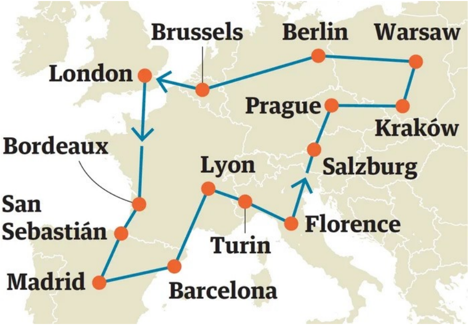
Exploration



$t = 7$

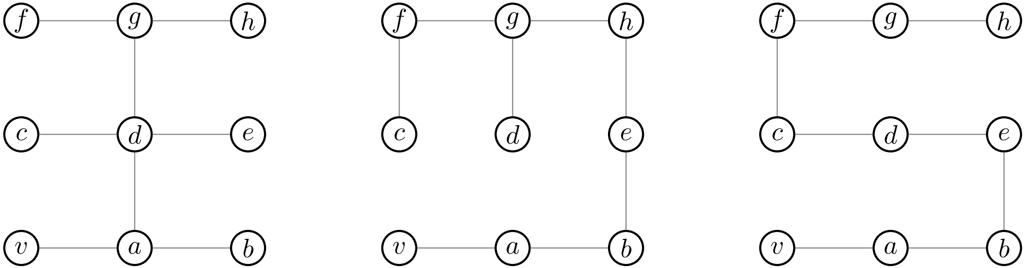
Temporal Exploration

Task: visit all European cities by train before your interrail pass expires. However, you can only travel between cities A and B at a given time if there is a train scheduled at that time.



Just like the *Graphical Travelling Salesperson Problem* but you can only use edges at **certain times**.

Always connected temporal graph

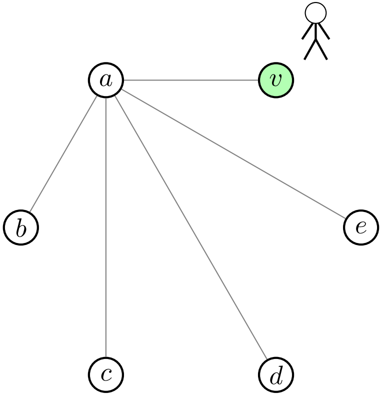


$t = 1$ $t = 2$ $t = 3$

Temporal Exploration

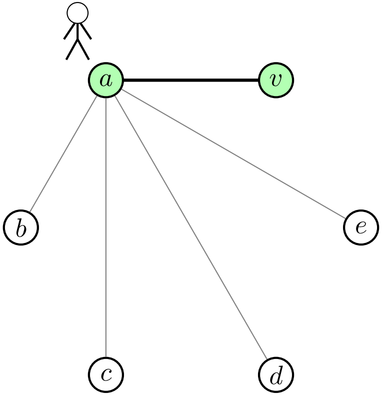
- Michail and Spirakis, 2014
  - no $(2-\epsilon)$-approximation unless P=NP.
  - Always connected temporal graphs are explorable in n^2 steps.
- Erlebach, Hoffmann and Kammer, 2015
  - No $n^{1-\epsilon}$-approximation unless P=NP.
  - There exists an always connected temporal graph that needs $\Omega(n^2)$ time to explore.
  - Graphs with *regularly present/probabilistic* edges can be explored in at most $O(m)$ in expectation, where m is the number of edges of the underlying graph.
  - some restricted classes can be explored in $o(n^2)$ time.
- Adamson, Gusev, Malyshev, Zamaraev 2022
  - Improved exploration times for restricted classes.
- Many other works looking at
  - Other notions of exploration (crossing several edges at once etc).
  - Temporal graphs where the difference between snapshots is small (k-deficient).
  - NP-Hardness for even more restricted classes.

Exploring temporal stars



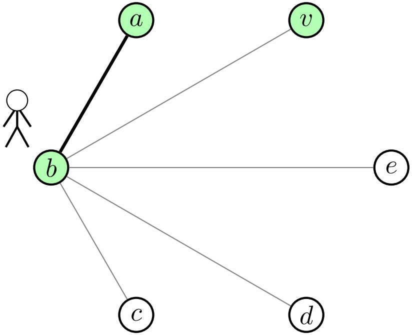
$t = 0$

Exploring temporal stars



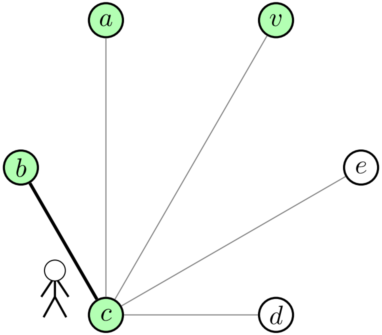
$t = 1$

Exploring temporal stars



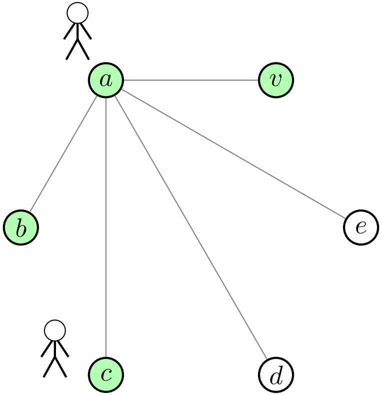
$t = 2$

Exploring temporal stars



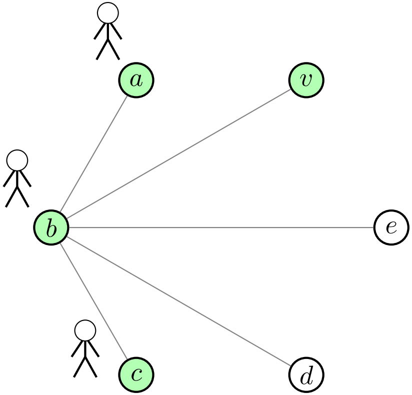
$t = 3$

Exploring temporal stars



$t = 4$

Exploring temporal stars



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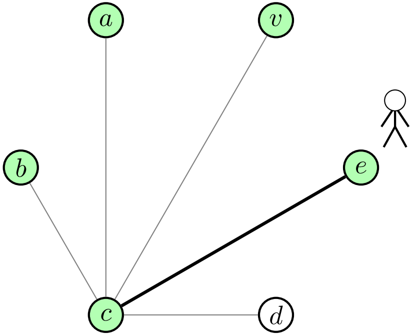
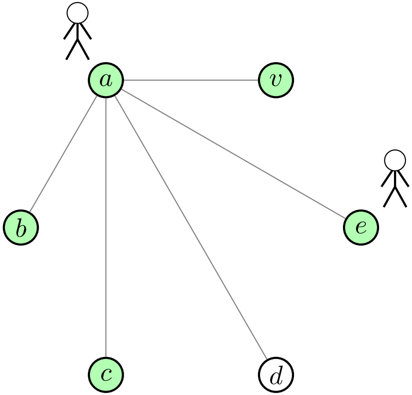
Exploring temporal stars



$t = 6$

Exploring temporal stars



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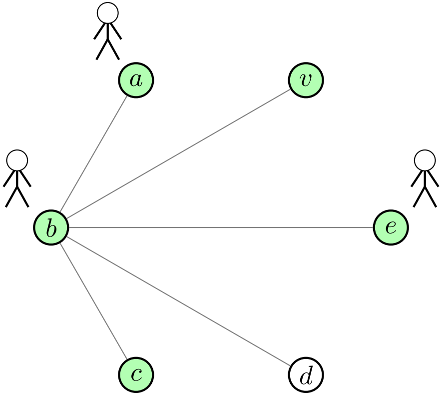
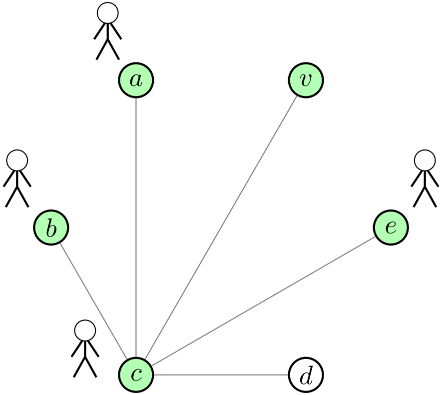
Exploring temporal stars



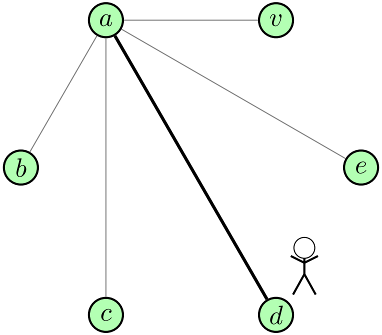
$t = 8$

Exploring temporal stars



$t = 9$

Exploring temporal stars



$t = 10$

Random Spanning Tree temporal graph

Definition (Random Spanning Tree (RST) temporal graph)

Let $\mathcal{T}$ be a set of trees with vertex set $[n]$, and let μ be a probability distribution on $\mathcal{T}$. We call the pair $(\mathcal{T}, \mu)$ a Random Spanning Tree model. A *Random Spanning Tree* (RST) temporal graph is a temporal graph $\mathcal{G} = (G_i)_{i \in \mathbb{N}}$ such that each G_i is a tree independently drawn from $\mathcal{T}$ according to μ. Abusing notation, we will write $\mathcal{G} \sim (\mathcal{T}, \mu)$.

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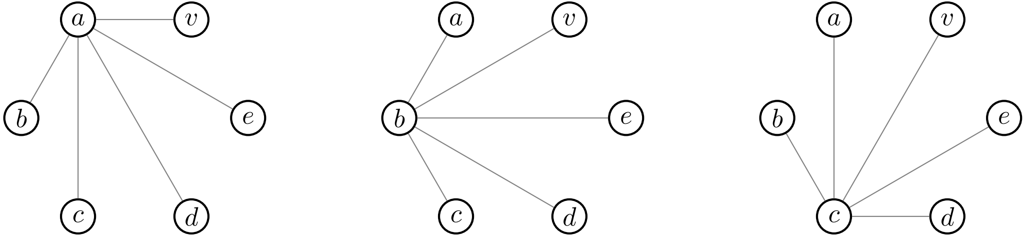

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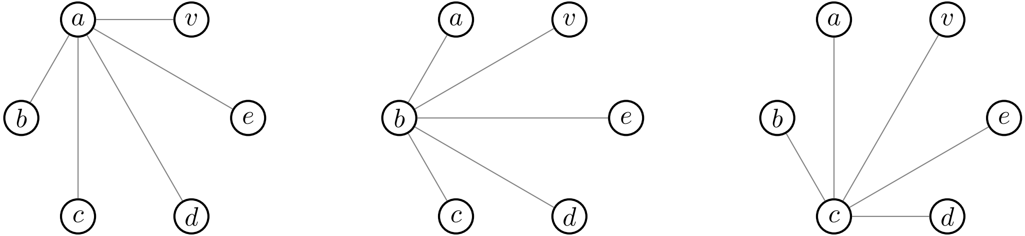
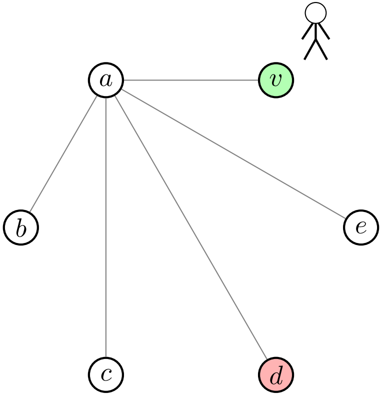


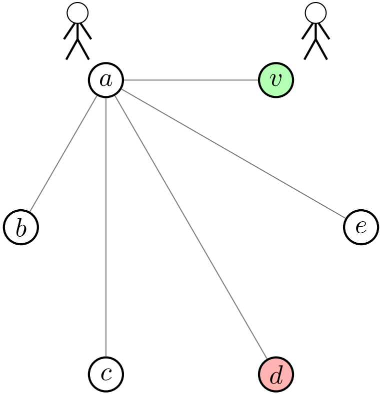
μ : 1/3 1/3 1/3

Exploring random stars



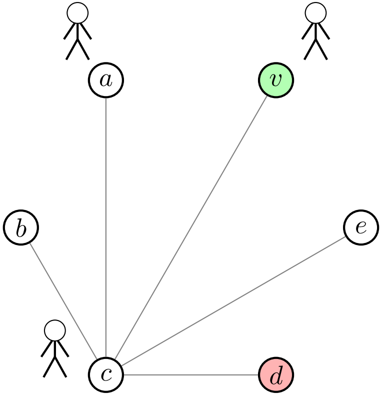
$t = 0$

Exploring random stars



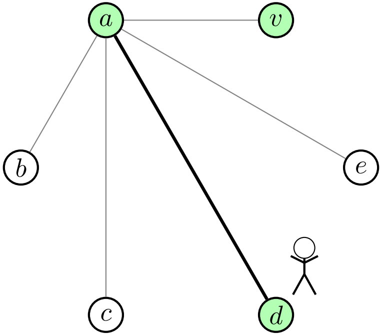
$t = 1$

Exploring random stars



$t = 2$

Exploring random stars



$t = 3$

Exploration lower bound

Theorem

Let $\mathcal{S}$ be a set of k stars on vertex set $[n]$, and let $\mathcal{G} \sim (\mathcal{S}, unif)$. Then

$$\mathbf{E}\left[\,\mathrm{TEXP}(\mathcal{G})\,\right] \in \Theta(\sqrt{k} \cdot (n-k) + k \log k).$$

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Corollary

Any algorithm that solves temporal exploration on arbitrary input distributions has a worst-case runtime of $\Omega(n^{3/2})$.

Exploration upper bound

Theorem (Exploration upper bound)

Let $\mathcal{T}$ *be a set of trees on vertex set* $[n]$*,* μ *be a probability distribution on* $\mathcal{T}$*, and* $\mathcal{G} \sim (\mathcal{T}, \mu)$*. Then*

$$\mathbf{Pr}[\mathrm{TEXP}(\mathcal{G}) \le 200000 \cdot n^{3/2}] \ge 1 - e^{-n}.$$

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Definition (Close vertices)

Let $u, w \subseteq [n]$, $t \in \mathbb{N}$, $p \in [0, 1]$. We say that w *is (t, p)-close to u in $(\mathcal{T}, \mu)$* if

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Close vertices

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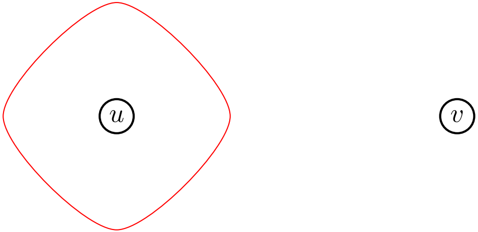

- Red circle = reachability set of u after t steps.

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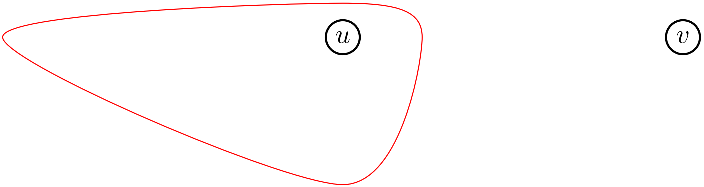

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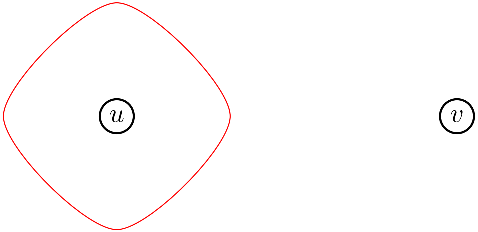

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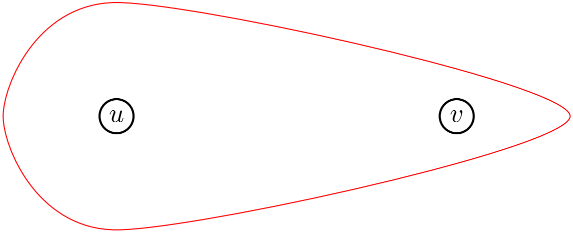

- Red circle = reachability set of u after t steps.
- Probability of v being inside the red circle of at least p.

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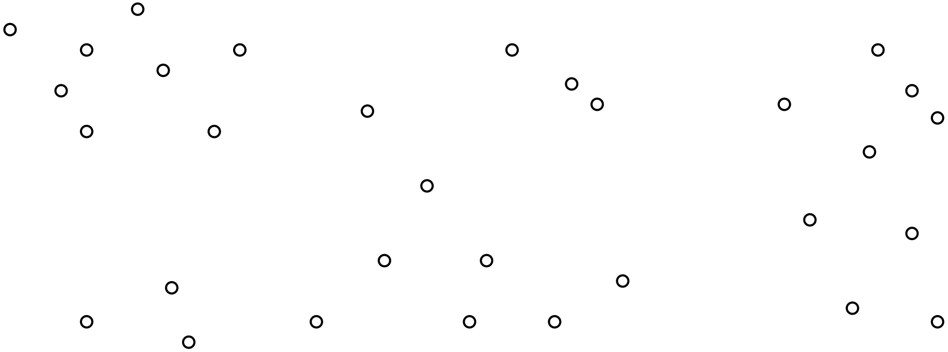
Theorem (Close vertices)

Let $n \in \mathbb{N}$ be sufficiently large, $\mathcal{T}$ be a set of trees on vertex set $[n]$, and μ be a probability distribution on $\mathcal{T}$. Then, for every vertex $v \in [n]$ there are at least $\sqrt{n}$ vertices that are $(700\sqrt{n}, 1/9)$-close to v in $(\mathcal{T}, \mu)$.

Close vertices -> Exploration



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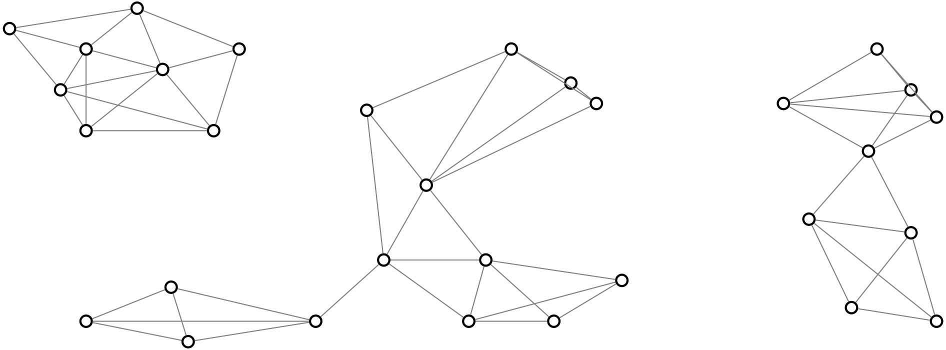
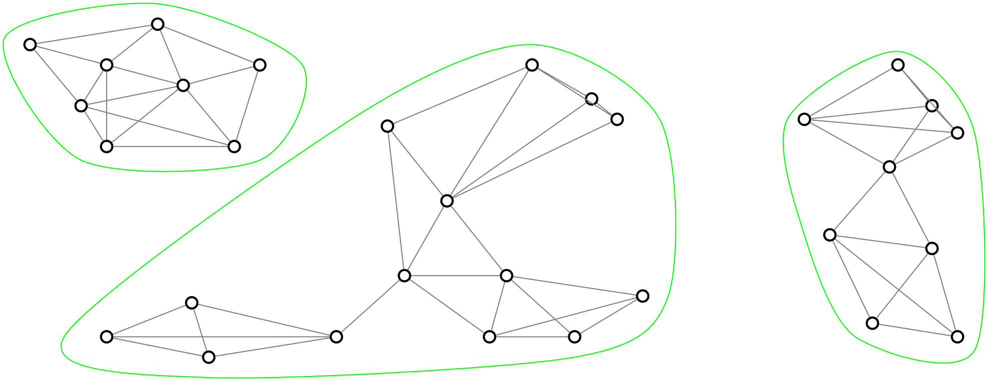


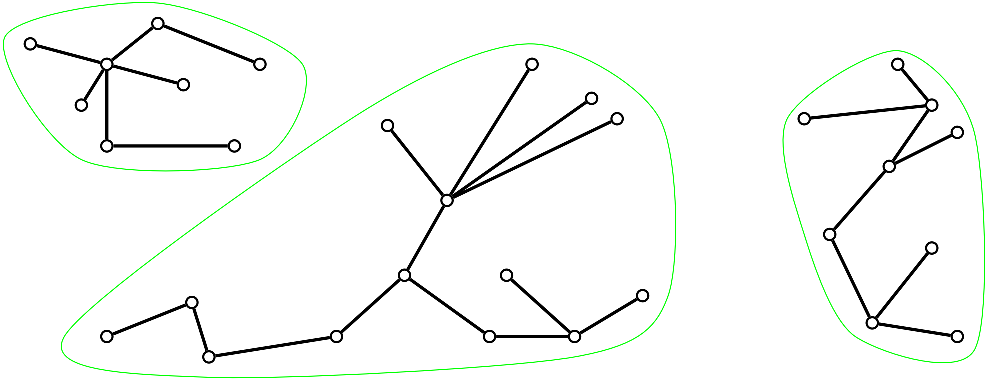
Connect all pairs of close vertices.

Close vertices -> Exploration



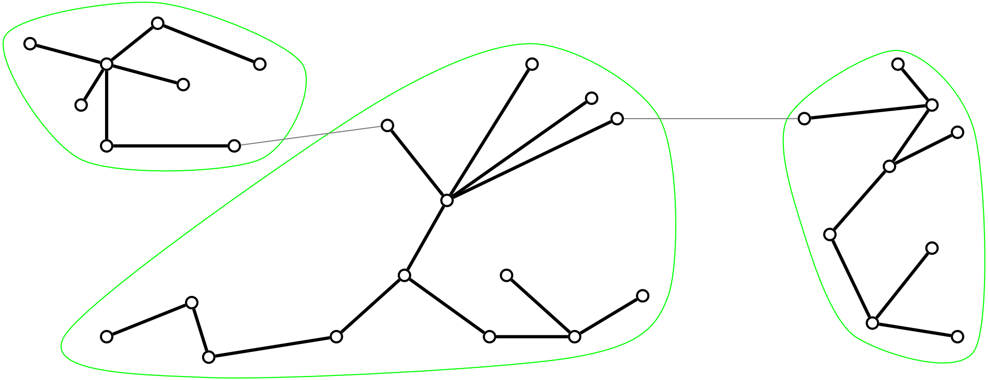
Each component is of size at least $\sqrt{n}$.

Close vertices -> Exploration



Build a spanning forest from those connections.

Close vertices -> Exploration



Connect the components arbitrarily to a tree.

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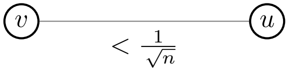
Close vertices proof

We aim to show that v has a $(c\sqrt{n}, p)$-close vertex u.

Close vertices proof

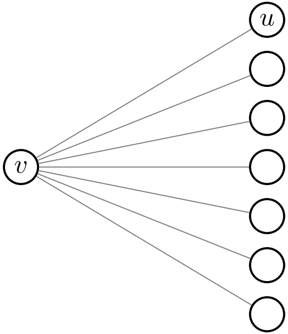
v has to have an edge in every timestep.

Close vertices proof



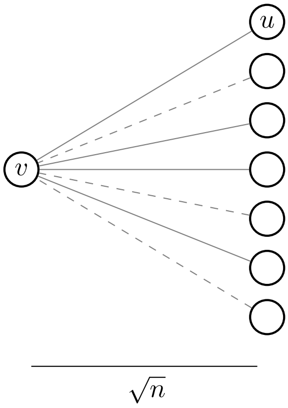
If any edge has a probability of probability at least $\frac{1}{\sqrt{n}}$ we are done.

Close vertices proof



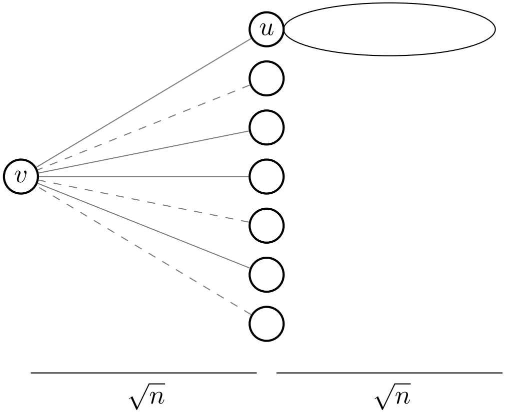
Otherwise it has to have a lot of potential neighbors.

Close vertices proof



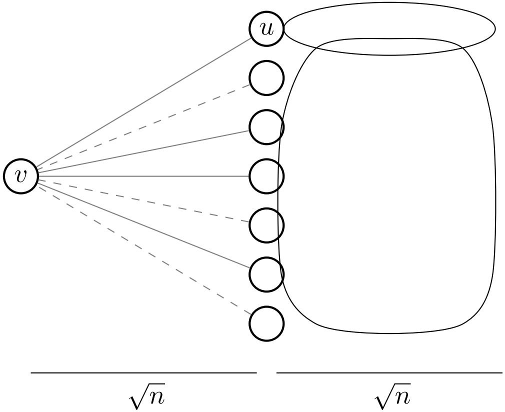
We look at $\sim \sqrt{n}$ timesteps and track the neighbors.

Close vertices proof



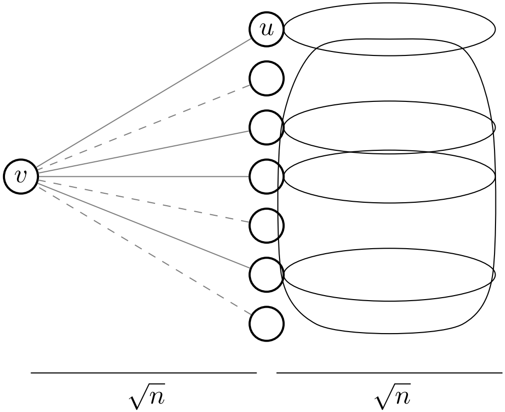
Look at the next $\sim \sqrt{n}$ timesteps and track the vertices u can reach.

Close vertices proof



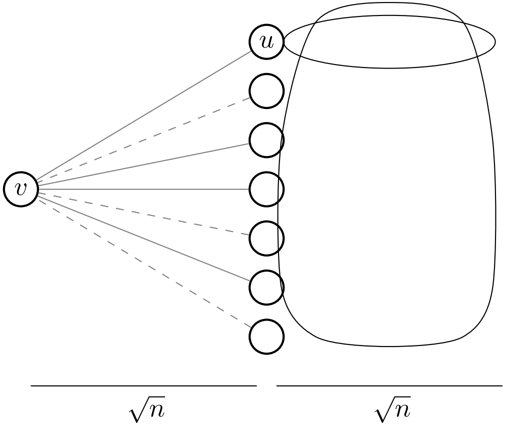
Case 1: there is no vertex that overlaps much with the sets of the other reached vertices.

Close vertices proof



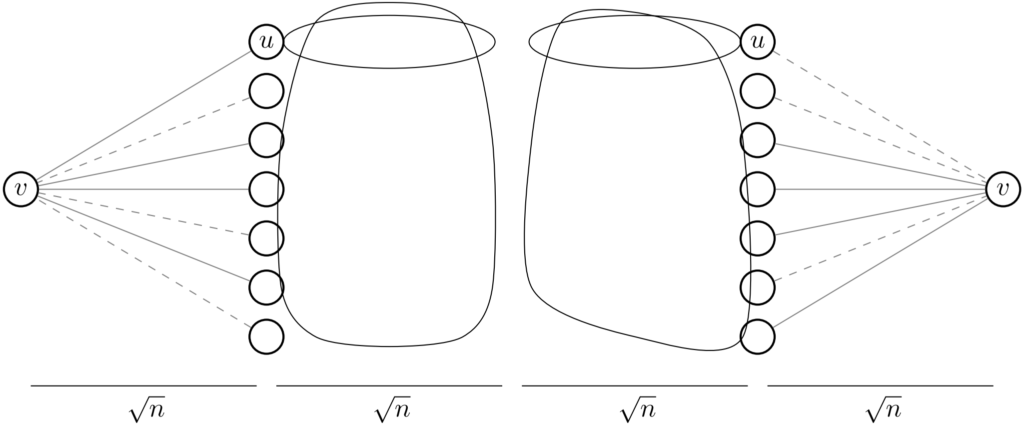
Then all these sets are mostly disjoint and sum to more than n which is a contradiction.

Close vertices proof



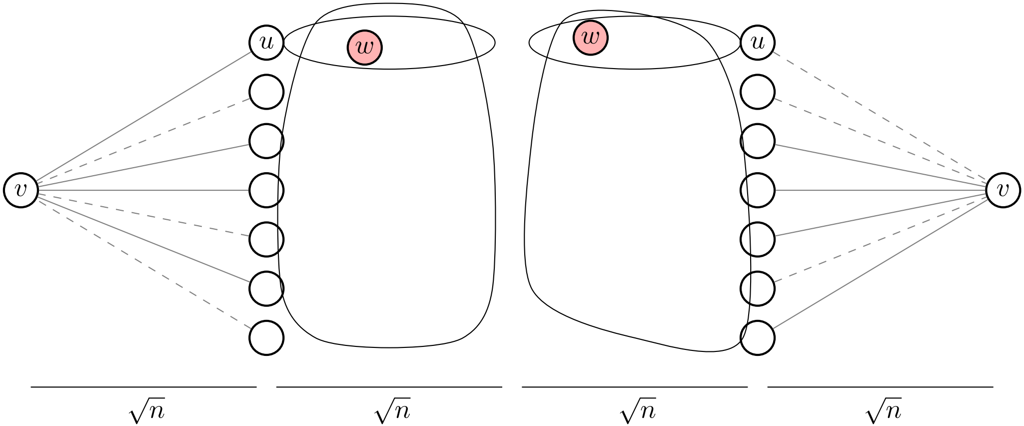
Case 2: A vertex u that is reached by v shares most of its reached vertices with others.

Close vertices proof



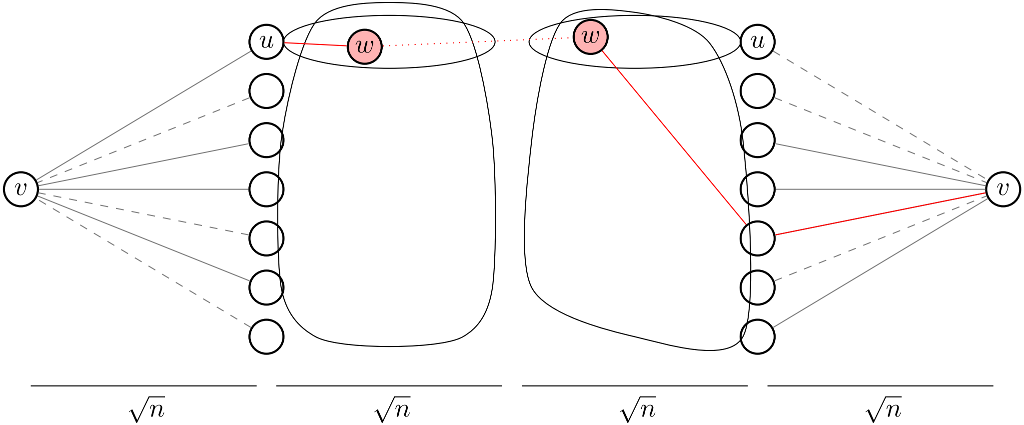
We run the same process backwards from v at a later time.

Close vertices proof



By birthday paradox, there is with constant probability a vertex w reached by u and another reached vertex in both the forward and backward run.

Close vertices proof



That implies a path with constant probability from u to v, so they are close to each other.

Open problems

- Exploration in time $O(\sqrt{d} \cdot n)$ on distributions with maximum degree d?
- Is greedily visiting the closest unvisited vertex asymptotically optimal?
- Linear exploration of uniform spanning trees of an arbitrary base graph?
- Modelling the temporal graph as a Markov chain.