Dismountability in Temporal Cliques Revisited

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July 7th, 2025

Temporal (happy) graphs

$\mathcal{G} = (V, E, \lambda)$, where $\lambda : E \to 2^{\mathbb{N}}$ assigns *time labels* to edges.

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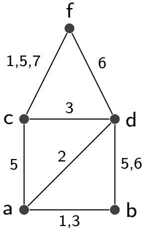

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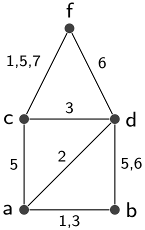

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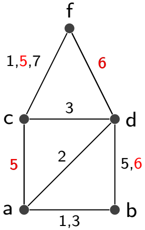

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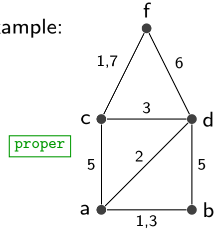

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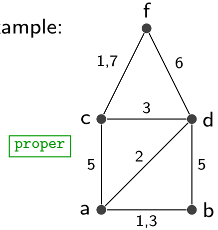

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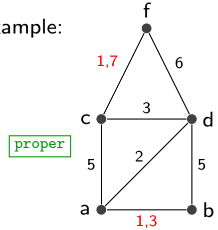

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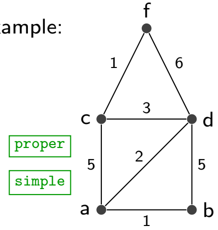

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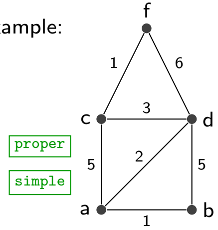

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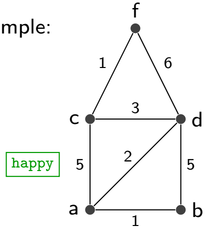

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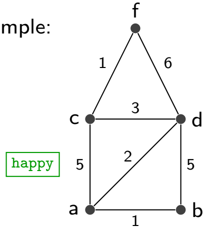

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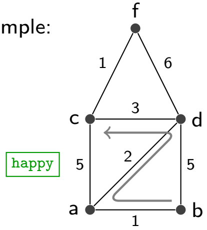

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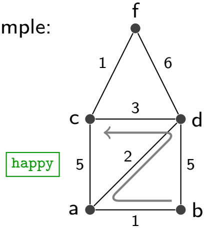

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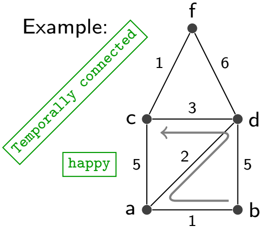

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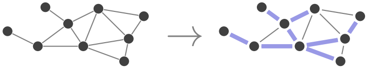

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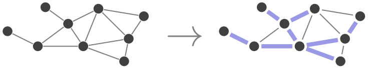

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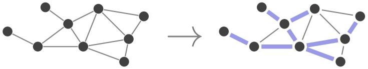

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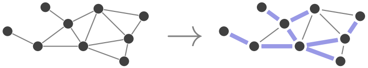

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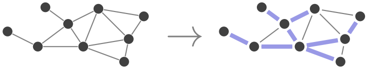

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Question: Is there any guarantee on the size?

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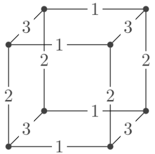

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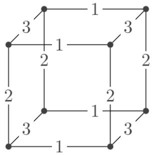

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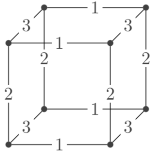

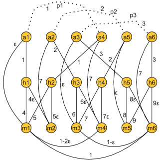

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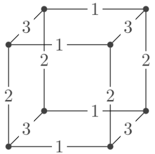

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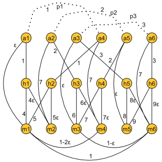

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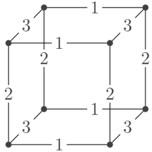

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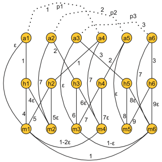

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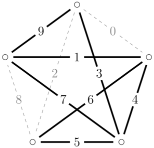

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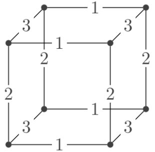

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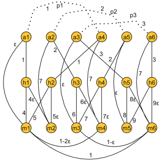

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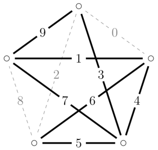

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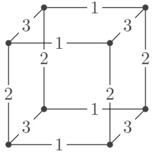

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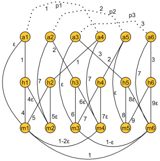

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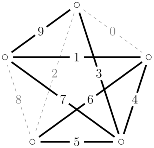
  In fact (this talk), dismountability is all you need!



Dismountability

(1-hop) Dismountability

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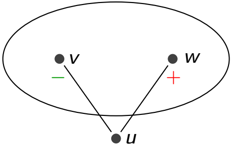

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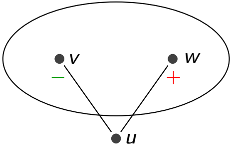

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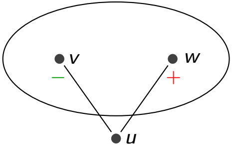

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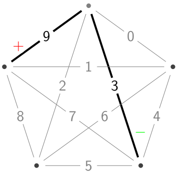

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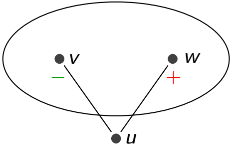
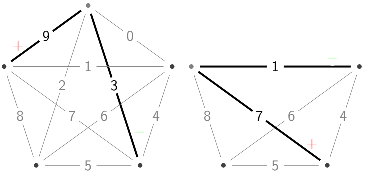


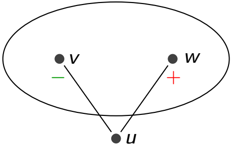
Then spanner($\mathcal{G}$) := spanner($\mathcal{G}[V \backslash u]$) + uv + uw → Recurse.



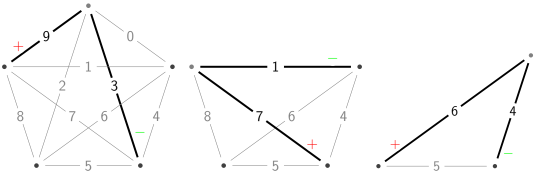
(1-hop) Dismountability

Find a node $\boldsymbol{u}$ s.t. :

- uv = minimum edge of some $\boldsymbol{v}$ (denoted $e^{-}(v)$)
- uw = maximum edge of some $\boldsymbol{w}$ (denoted $e^{+}(w)$)



Then $\texttt{spanner}(\mathcal{G}) := \texttt{spanner}(\mathcal{G}[V \backslash u]) + uv + uw \rightarrow$ Recurse.



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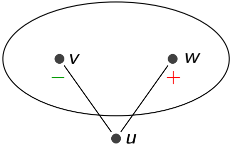
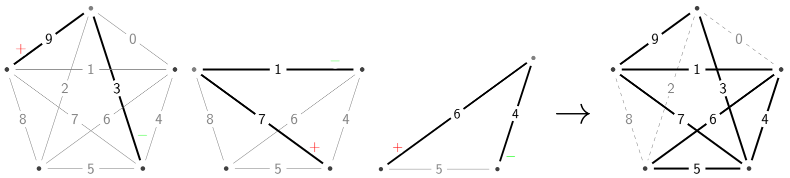


Then $\texttt{spanner}(\mathcal{G}) := \texttt{spanner}(\mathcal{G}[V \setminus u]) + uv + uw \rightarrow$ Recurse.



If recursively applicable, it yields a spanner of size $2n - 3$.

(1-hop) Dismountability

Find a node $\boldsymbol{u}$ s.t. :

- uv = minimum edge of some $\boldsymbol{v}$ (denoted $e^{-}(v)$)
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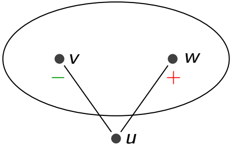
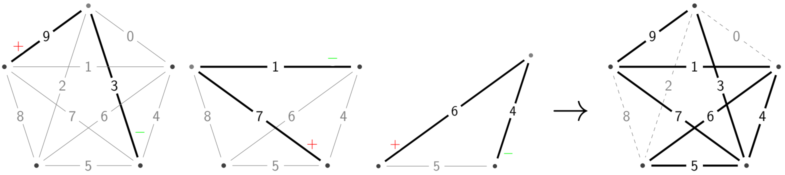


Then $\texttt{spanner}(\mathcal{G}) := \texttt{spanner}(\mathcal{G}[V \backslash u]) + uv + uw \rightarrow$ Recurse.

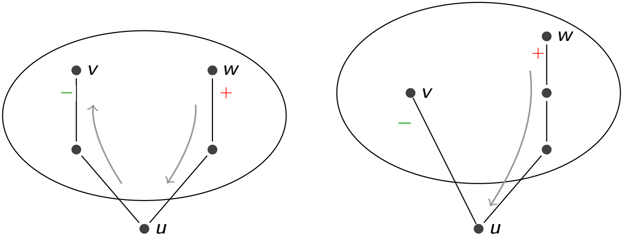


If recursively applicable, it yields a spanner of size $2n - 3$.

Unfortunately, it is not always applicable.

Relaxed version: k-hop dismountability

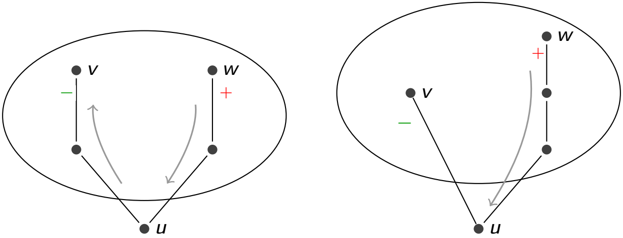
Temporal paths $u \rightsquigarrow v$ ending at $e^-(v)$ and $w \rightsquigarrow u$ starting at $e^+(w)$



(a) Example of 2-hop dismountable (b) Example of 3-hop dismountable

Relaxed version: k-hop dismountability

Temporal paths $u \rightsquigarrow v$ ending at $e^-(v)$ and $w \rightsquigarrow u$ starting at $e^+(w)$



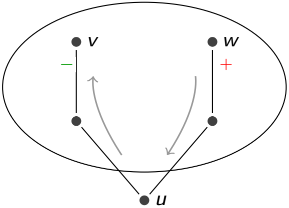
(a) Example of 2-hop dismountable (b) Example of 3-hop dismountable

→ Select both paths in the spanner
→ recurse! (in $\mathcal{G} \setminus u$)

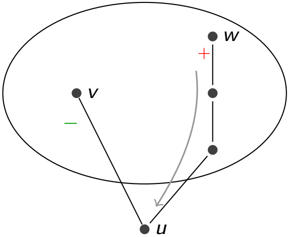
If it is recursively applicable for some $k \in O(1)$, we obtain an $O(n)$ spanner.

Relaxed version: k-hop dismountability

Temporal paths $u \rightsquigarrow v$ ending at $e^{-}(v)$ and $w \rightsquigarrow u$ starting at $e^{+}(w)$



(a) Example of 2-hop dismountable



(b) Example of 3-hop dismountable

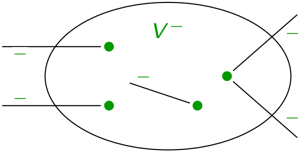
$\rightarrow$ Select both paths in the spanner
$\rightarrow$ recurse! (in $\mathcal{G} \setminus u$)

If it is recursively applicable for some $k \in O(1)$, we obtain an $O(n)$ spanner.

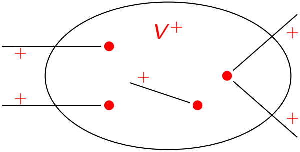
Again, not always applicable, but...

The absence of k-hop dismountable vertices gives rise to an interesting structure.

Non 1-hop dismountable cliques

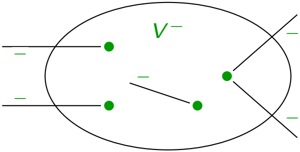


$V^- = \{u \in V : uv = e^-(v) \text{ for some } v\}$



$V^+ = \{u \in V : uv = e^+(v) \text{ for some } v\}$

Non 1-hop dismountable cliques



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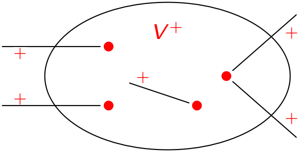
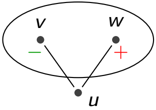


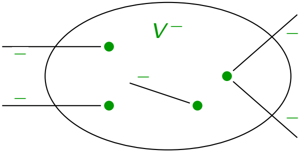
$V^+ = \{u \in V : uv = e^+(v) \text{ for some } v\}$

If u belongs to both V^- and V^+ then it is 1-hop dismountable!

$\Longrightarrow$ In non 1-hop dismountable cliques $V^- \cap V^+ = \emptyset$.



Non 1-hop dismountable cliques



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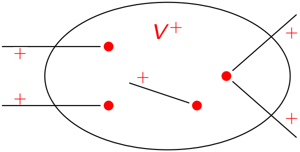
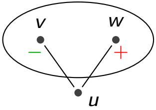


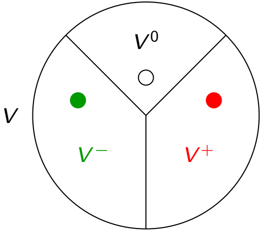
$V^+ = \{u \in V : uv = e^+(v) \text{ for some } v\}$

If u belongs to both V^- and V^+ then it is 1-hop dismountable!

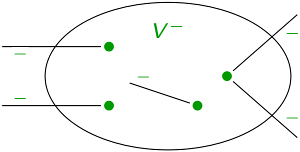
$\implies$ In non 1-hop dismountable cliques $V^- \cap V^+ = \emptyset$.



If the clique is **non 1-hop dismountable**, then V can be partitioned into V^-, V^+, and $V^0 = V \setminus (V^- \cup V^+)$.



Non 1-hop dismountable cliques



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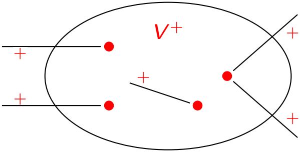
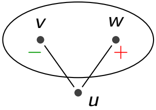


$V^+ = \{u \in V : uv = e^+(v) \text{ for some } v\}$

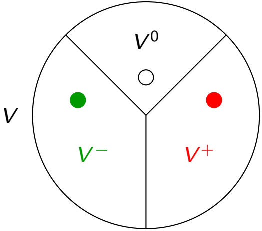
If u belongs to both V^- and V^+ then it is 1-hop dismountable!

$\implies$ In non 1-hop dismountable cliques $V^- \cap V^+ = \emptyset$.



If the clique is **non 1-hop dismountable**, then V can be partitioned into V^-, V^+, and $V^0 = V \setminus (V^- \cup V^+)$.

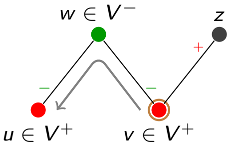
Alternatively, V^0 is the set of vertices that do not *receive* any minimum or maximum edge from any other vertex.

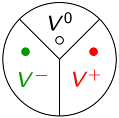


Non $\{1,2\}$-hop dismountable cliques

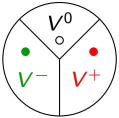
If the minimum edge of two or more vertices in V^+ goes to the same vertex in V^- then 2-hop dismountable.

The same holds for maximum edges of vertices in V^-.



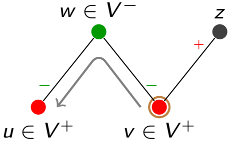


Non $\{1,2\}$-hop dismountable cliques



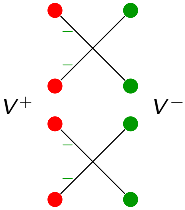
If the minimum edge of two or more vertices in V^+ goes to the same vertex in V^- then 2-hop dismountable.

The same holds for maximum edges of vertices in V^-.



Consequences for non $\{1,2\}$-hop dismountable cliques:

- The edges $\{e^-(v) : v \in V^+\}$ form a *matching*.



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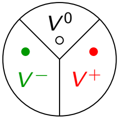

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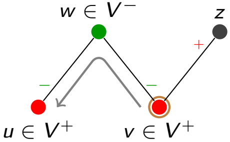

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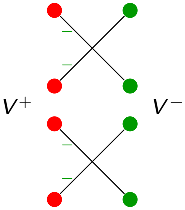

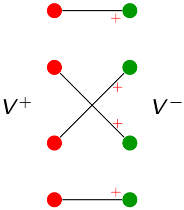

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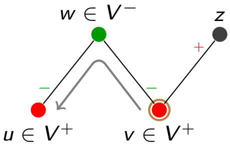
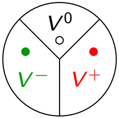
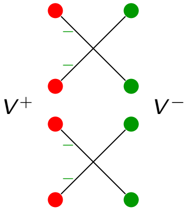
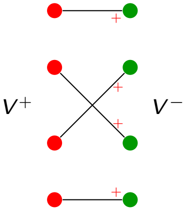




Consequences for non $\{1,2\}$-hop dismountable cliques:

- The edges $\{e^-(v) : v \in V^+\}$ form a *matching*.
- The edges $\{e^+(v) : v \in V^-\}$ form a *matching*.
- V^- and V^+ are of *equal size*.





Non $\{1,2\}$-hop dismountable cliques

If the minimum edge of two or more vertices in V^+ goes to the same vertex in V^- then 2-hop dismountable.

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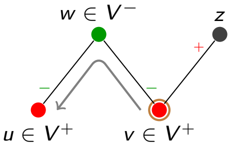

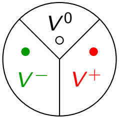

Consequences for non $\{1,2\}$-hop dismountable cliques:

- The edges $\{e^-(v) : v \in V^+\}$ form a *matching*.
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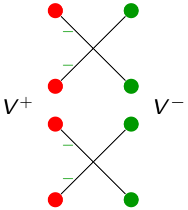
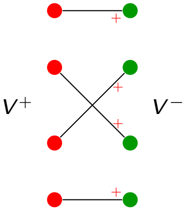
Question: Where are the vertices of V^0?

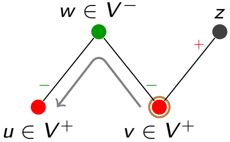


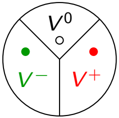


Non $\{1,2\}$-hop dismountable cliques

If the minimum edge of two or more vertices in V^+ goes to the same vertex in V^- then 2-hop dismountable.

The same holds for maximum edges of vertices in V^-.





Consequences for non $\{1,2\}$-hop dismountable cliques:

- The edges $\{e^-(v) : v \in V^+\}$ form a *matching*.
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- V^- and V^+ are of *equal size*.

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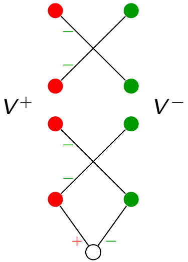
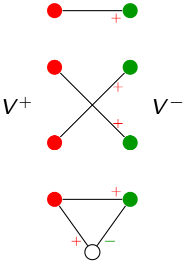
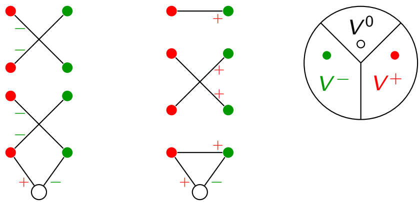
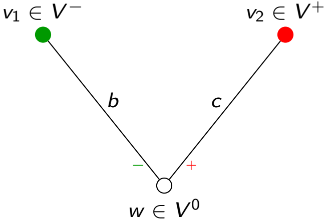




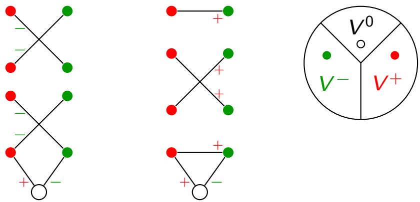
Non $\{1,2\}$-hop dismountable cliques



What about V^0 ?



Non $\{1,2\}$-hop dismountable cliques



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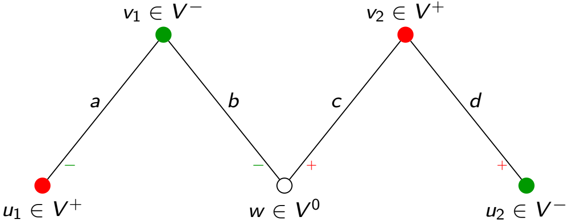
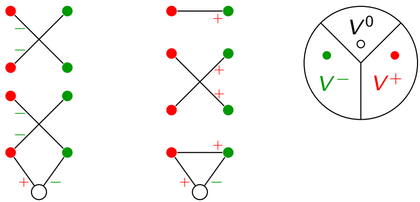


Non $\{1,2\}$-hop dismountable cliques



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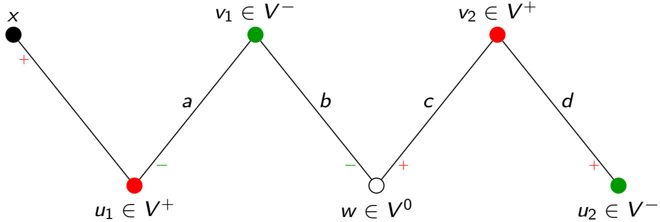
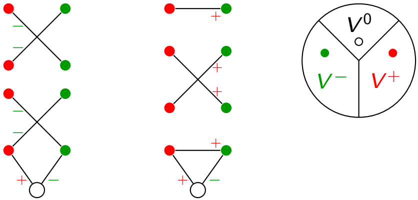


Non $\{1,2\}$-hop dismountable cliques



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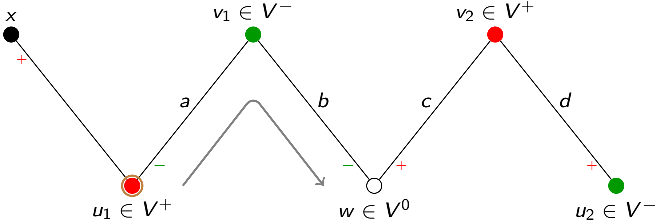
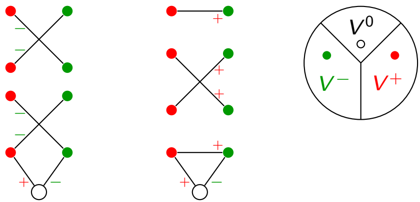


Non $\{1,2\}$-hop dismountable cliques



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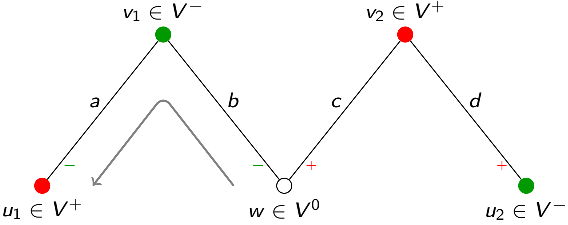
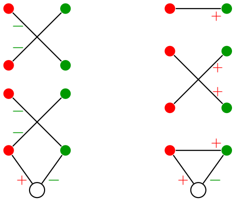
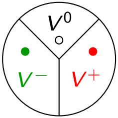
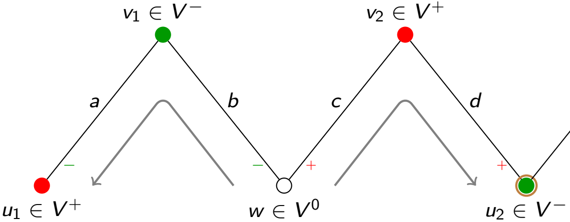


Non $\{1,2\}$-hop dismountable cliques





What about V^0 ?



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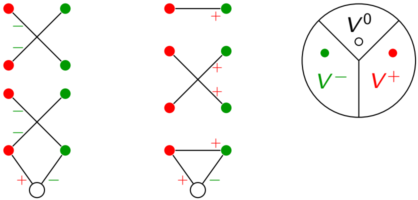

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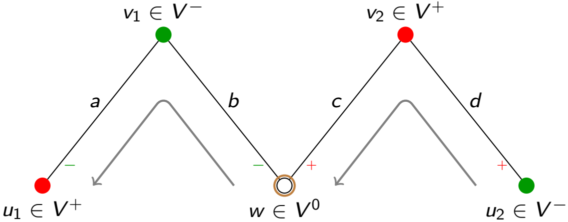

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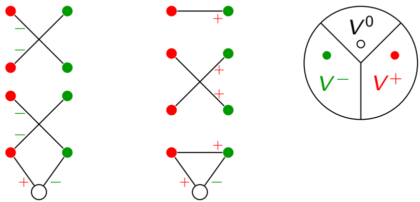

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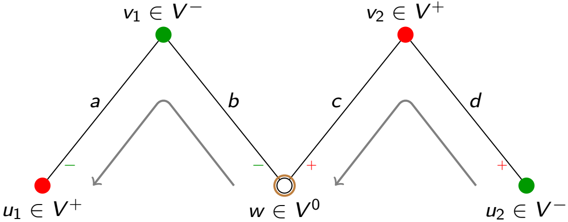
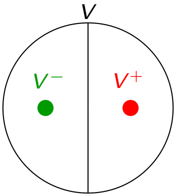
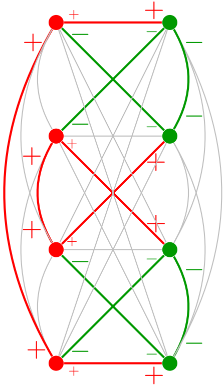


If $\mathcal{G}$ is non $\{1,2\}$-hop dismountable, then $\boldsymbol{V^0}$ **is empty**.

Summary of non $\{1,2\}$-hop dismountable cliques



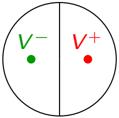


If $\mathcal{G}$ is non $\{1,2\}$-hop dismountable, then:

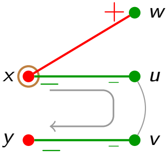
1. V^- and V^+ are the same size and form a **partition** of V.
2. The set $M^- := \{e^-(v) : v \in V^+\}$ is a **perfect matching**.
3. The set $M^+ := \{e^+(v) : v \in V^-\}$ is a **perfect matching**.

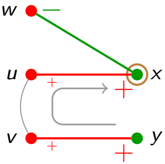
(Actually, if and only if)

Non $\{1,2,3\}$-hop dismountable cliques



A non $\{1,2\}$-hop dismountable clique is 3-hop dismountable if and only if we have such temporal paths:





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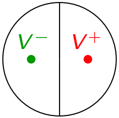

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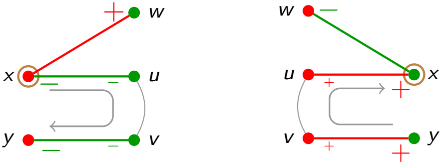
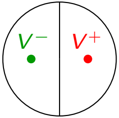


Theorem 3,7: $\mathcal{G}$ k-hop dismountable $\implies$ $\mathcal{G}$ $\{1,2,3\}$-hop dismountable

Non $\{1,2,3\}$-hop dismountable cliques



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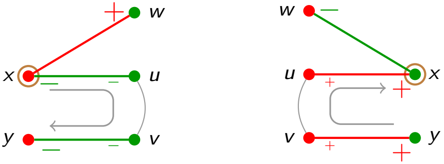

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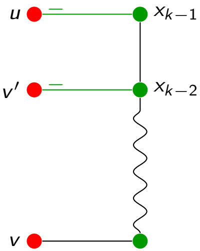
By contradiction:

$k \geq 4$ minimal

$v \in V^+$ can reach $e^-(u)$ in k-hop

through path $v, x_1, x_2, \ldots, x_{k-1}, u$



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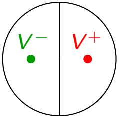

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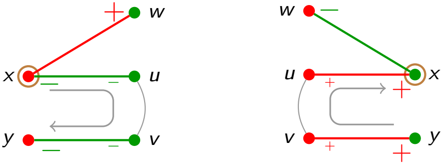

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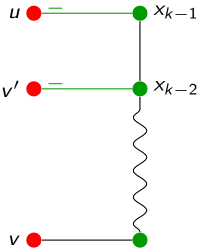

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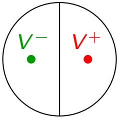

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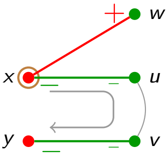

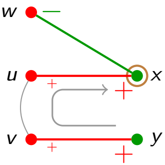

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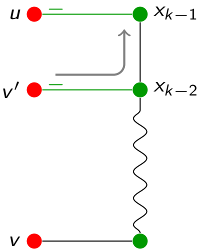

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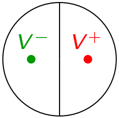

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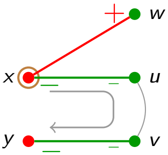

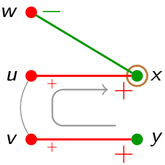

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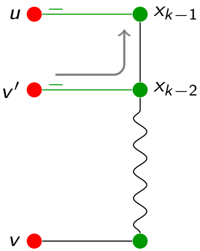

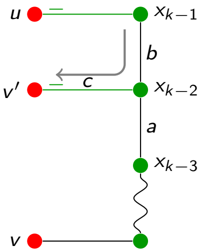

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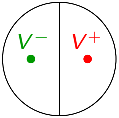

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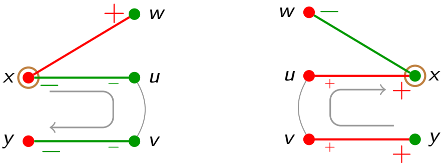

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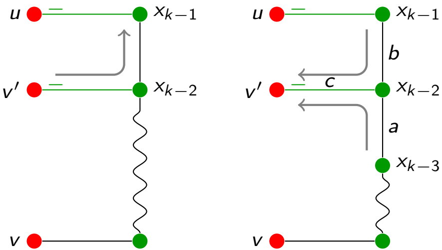

Consequences of Theorem 3.7

Theorem 3,7: $\mathcal{G}$ k-hop dismountable $\implies$ $\mathcal{G}$ $\{1, 2, 3\}$-hop dismountable

Consequences:

- We can stop the analysis at $k = 3$.

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- Any **minimal counter-example** to the existence of $4n$ spanners must have all the properties of non $\{1,2,3\}$-hop dismountable.

Consequences of Theorem 3.7

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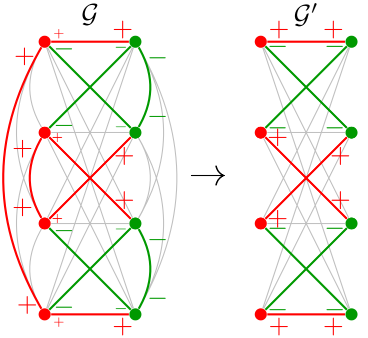
Consequences:

- We can stop the analysis at $k = 3$.
- Any **minimal counter-example** to the existence of $4n$ spanners must have all the properties of non $\{1,2,3\}$-hop dismountable.
- As far as $O(n)$ spanners are concerned, excluding $\{1,2\}$-hop dismountability is sufficient.

Why?

- Let $\mathcal{G}' \subseteq \mathcal{G}$ be the bipartite part between V^- and V^+.
- $\mathcal{G}'$ is **extremally matched** (reciprocal $-$ and $+$ edges)
- $\mathcal{G}' \in$ TC
- Any spanner of $\mathcal{G}'$ is a spanner of $\mathcal{G}$



Thm: Extremally matched bicliques admit $O(n)$ spanners **if and only if** temporal cliques admit $O(n)$ spanners.

($\implies$) [Casteigts, Peters, Schoeters, 2019]
($\impliedby$) [Angrick et al., 2024]

$O(n \log n)$ spanners using only dismountability

Let's work in extremally matched temporal bicliques

- We can add the two matching to the spanner (essentially free)
- Focus on preserving reachability **left** to **right** only (with matchings $\implies$ spanner TC)

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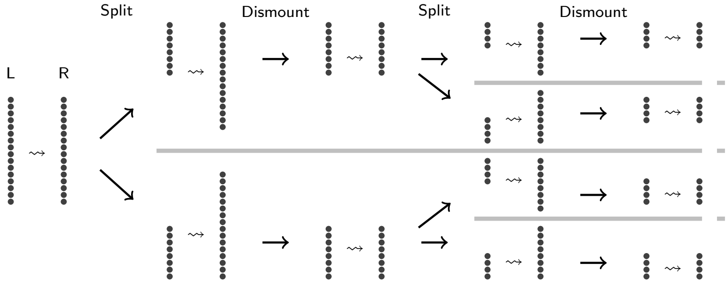
Theorem: $O(n \log n)$ spanners always exist [Casteigts, Peters, Shoeters, 2019]
↓ A much simpler proof by [Angrick et al., 2024]



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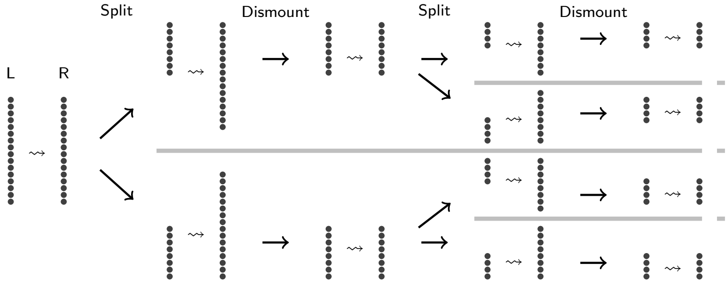

1. Split the work, achieving both halves of L to all of R separately.

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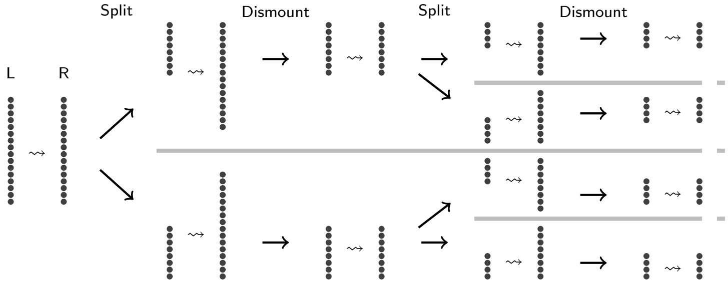
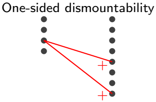


1. Split the work, achieving both halves of L to all of R separately.
2. Dismount vertices of R whose $+$ collide in L (pay two edges).



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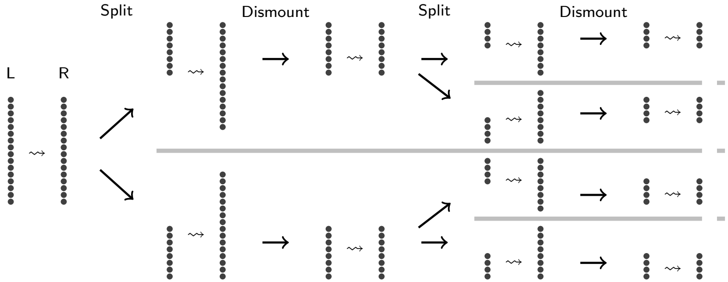
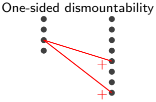


1. Split the work, achieving both halves of L to all of R separately.
2. Dismount vertices of R whose + collide in L (pay two edges).
3. Recurse.



$O(n \log n)$ spanners using only dismountability

Let's work in extremally matched temporal bicliques

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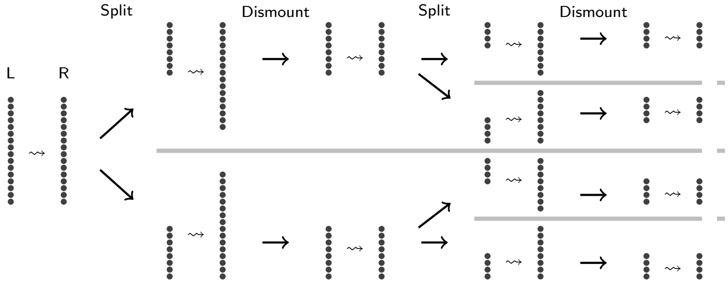

1. Split the work, achieving both halves of L to all of R separately.
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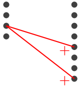
One-sided dismountability



$$cost(n) = 2 \cdot cost(n/2) + O(n)$$

By the Master's theorem for recurrences, the total cost is $\boxed{O(n \log n)}$ □

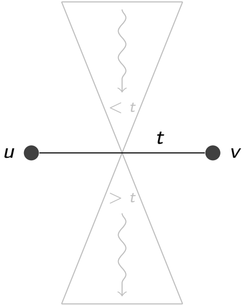
Pivotability and Recursively k-hop Dismountable

Pivotability and Recursively k-hop Dismountable

Let (e, t) be a temporal edge. If for every $w \in V$

- There exists a temporal path from w to e (ending with e), and
- There exists a temporal path from e to w (starting with e).

Then we say that the graph is *pivotable*.



$2n - 3$ spanner

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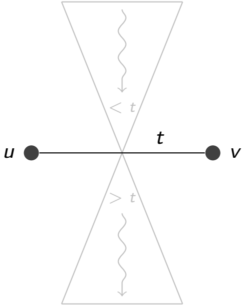

$2n - 3$ spanner

Theorem 5.2: Let $\mathcal{G}$ be a temporal clique. If $\mathcal{G}$ is recursively k-hop dismountable, then $\mathcal{G}$ is pivotable.

Summary

Theorem 3.10: $\mathcal{G}$ is non k-hop dismountable ($k \geq 3$) if and only if:

1. V^- and V^+ are the same size and form a **partition** of V.
2. Every edge between V^- and V^+ is later than all adjacent edges in E^- and earlier than all adjacent edges in E^+.
3. For every edge e within the part V^- (resp. V^+), the label of e cannot be between the labels of the two incident edges of M^- (resp. M^+).

- Any minimal counterexample to a $4n$ spanner must satisfy conditions 1, 2, and 3.
- Non $\{1, 2\}$-hop dismountable (conditions 1. and 2.) is sufficient to reduce the problem to **extremally matched bicliques**.
- Recursively k-hop dismountable $\implies$ pivotable.