Distance to Transitivity: New Parameters for Taming Reachability in Temporal Graphs

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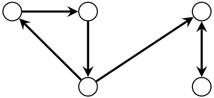

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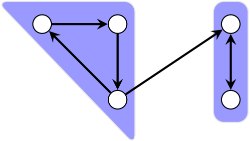

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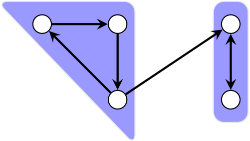

The unique partitioned into a a collection of sccs can be found in linear time. [Tarjan '72]

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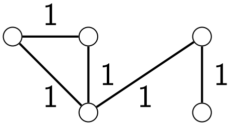

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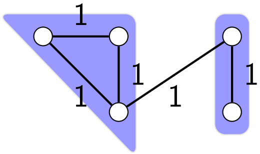

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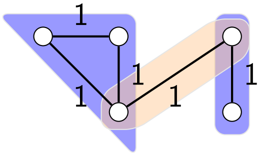

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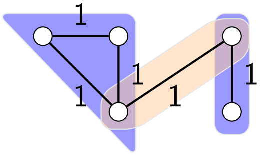

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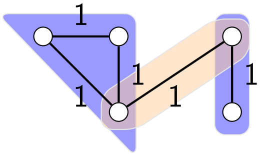

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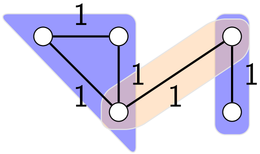

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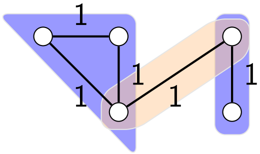

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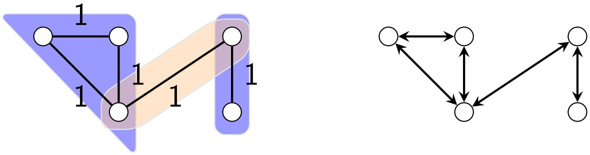

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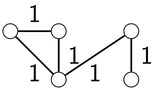
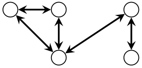
temporal		

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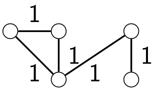
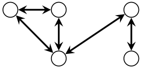
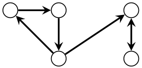
temporal		
static		

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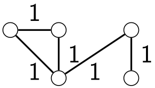
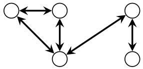
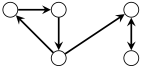
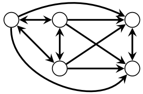
temporal		
static		

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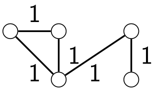
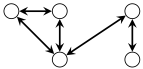
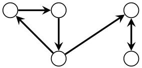
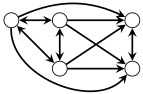
temporal		
static		

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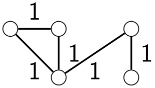
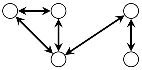
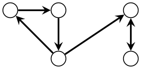
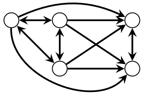
temporal		
static		

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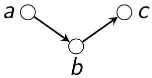

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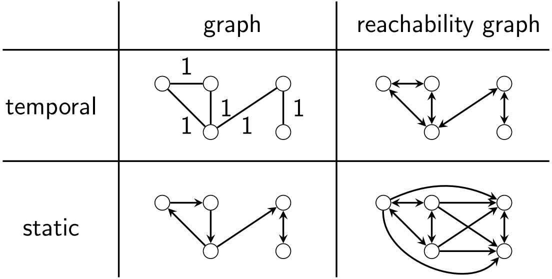

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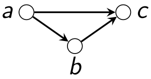

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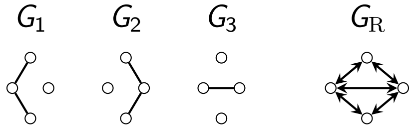

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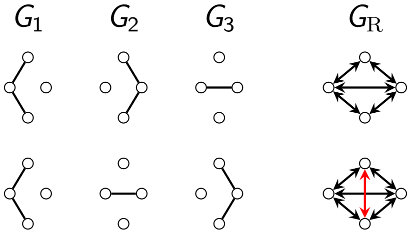

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TCC *admits a kernel of size* $\mathcal{O}(|M|^3)$, *where* M *is a given arc-modification set towards a transitive reachability graph.*

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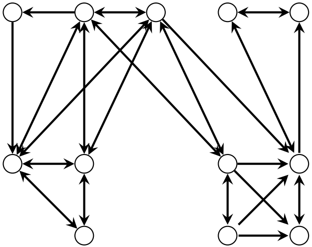
Recall: We search for bidirectional cliques



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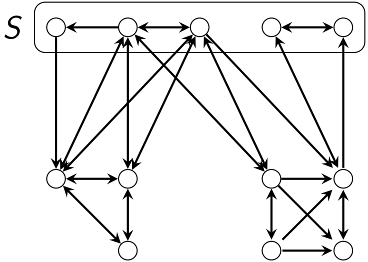

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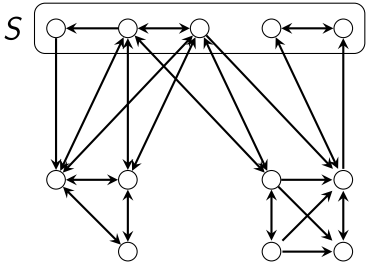

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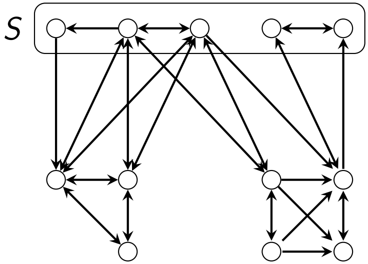

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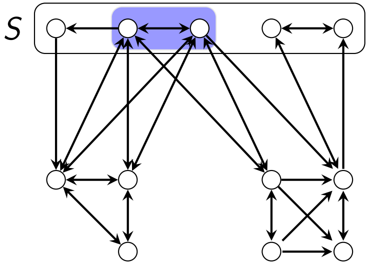

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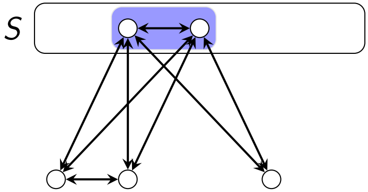

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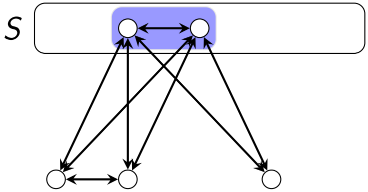

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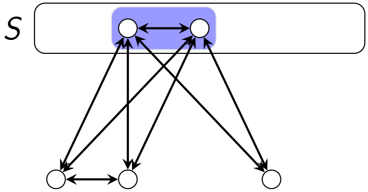

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TCC *does not admit a polynomial kernel for* δ_{vd}*, unless* $NP \subseteq coNP/poly$*.*

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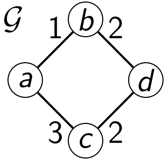

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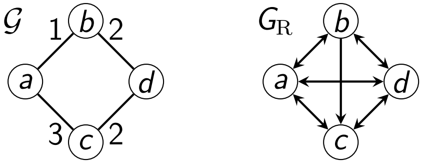

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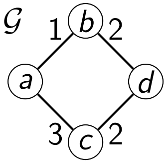

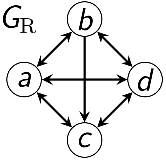

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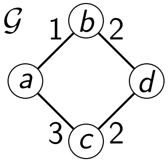

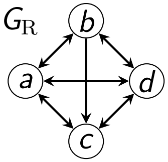

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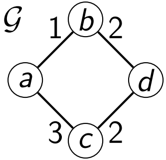

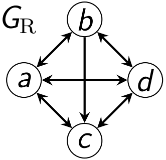

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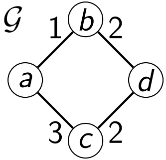

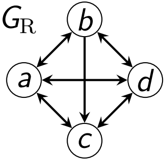

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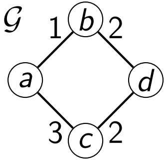

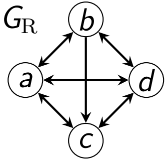

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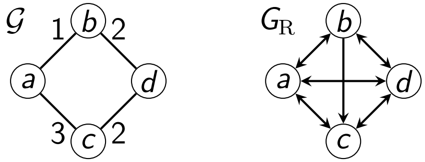

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Theorem

CTCC *is NP-hard and W[1]-hard when parameterized by k even if $\delta_{\mathrm{vd}} = \delta_{\mathrm{am}} = 1$.*

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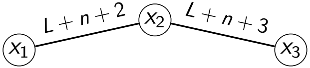

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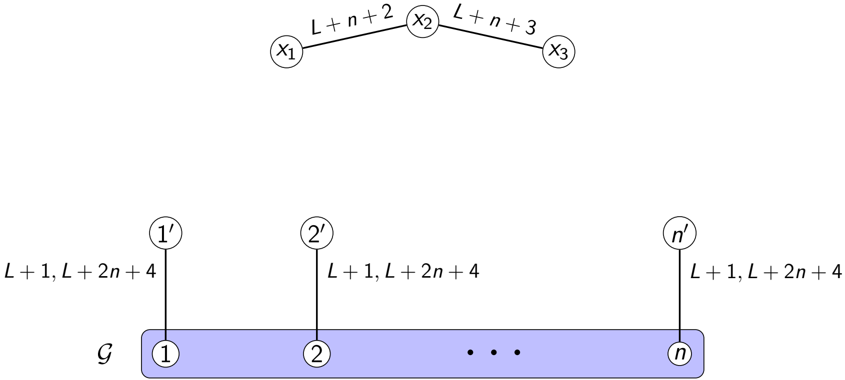

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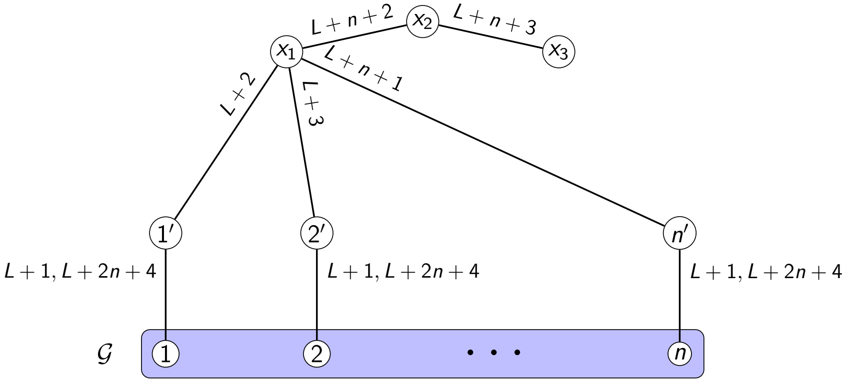

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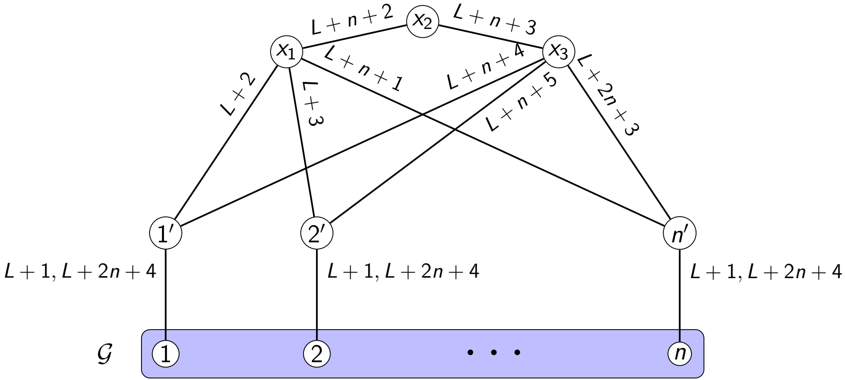

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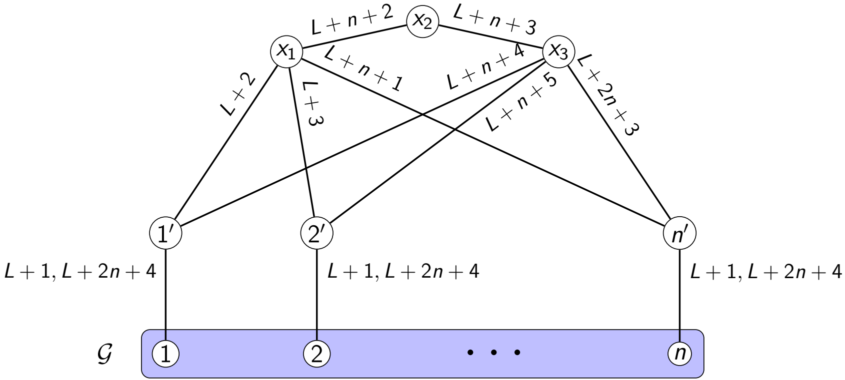

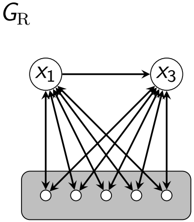

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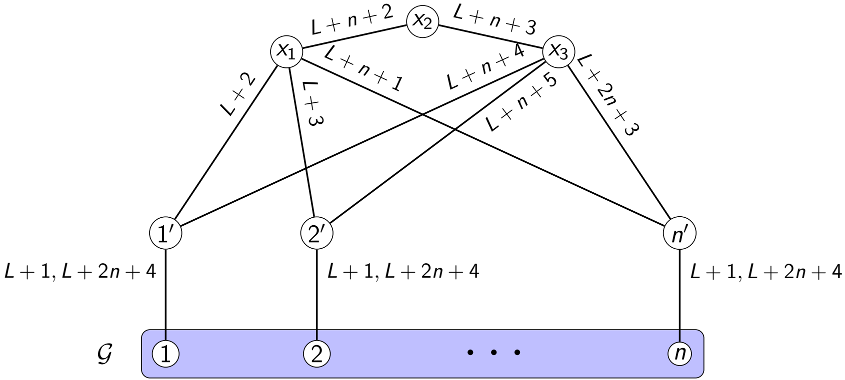

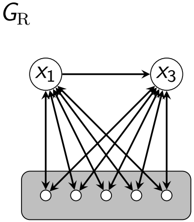

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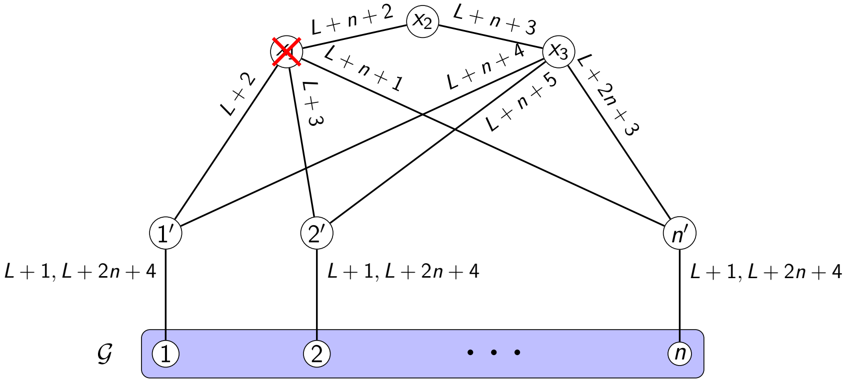

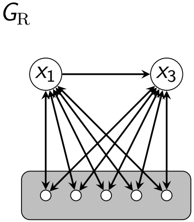

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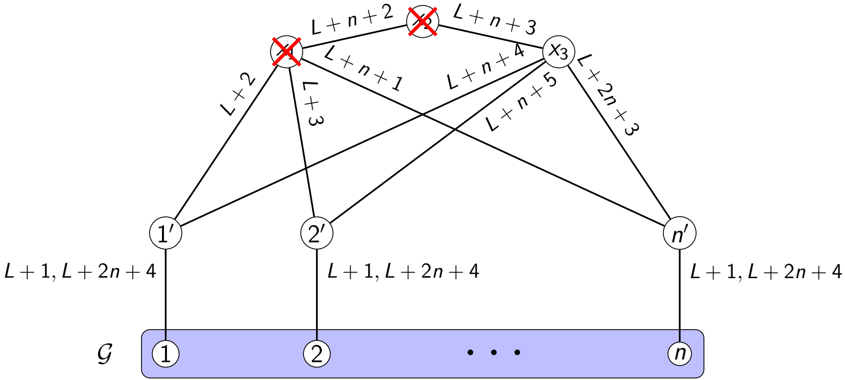

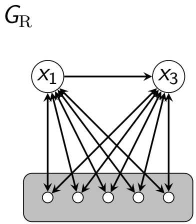

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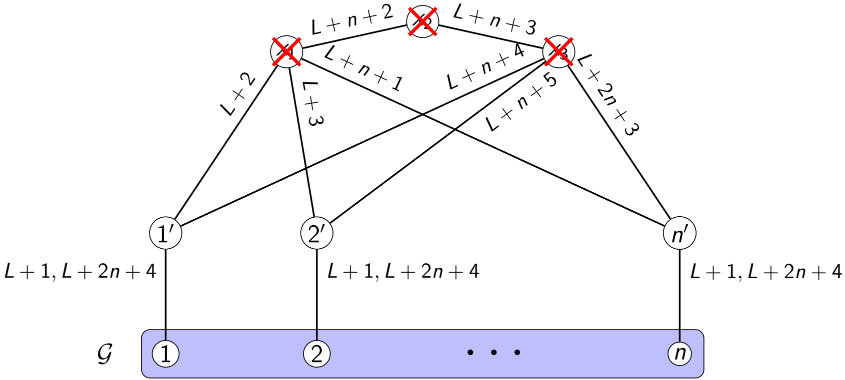

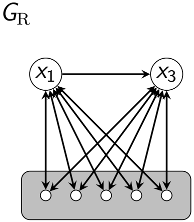

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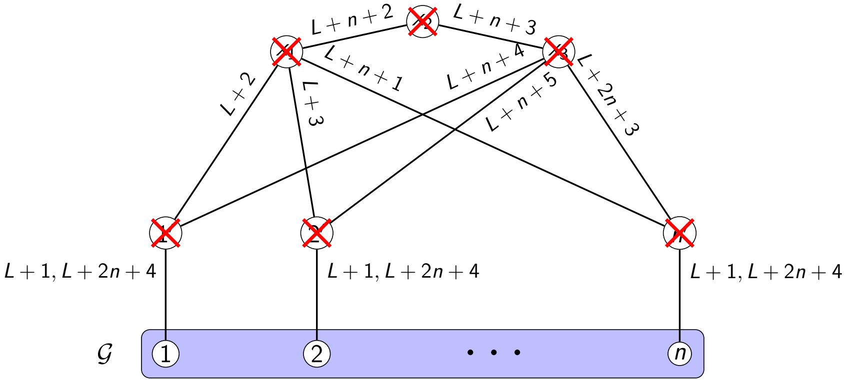

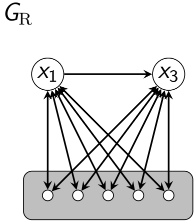

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