PARAMETERS FOR DENSE TEMPORAL GRAPHS

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July 5, 2024

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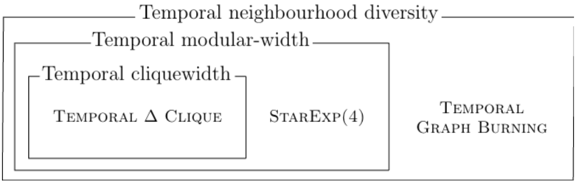
We like to do parameterised algorithmics on temporal graphs.

The parameters we had previously used for temporal graphs either requires sparseness of activity or don't work well.

We have started using a few generalisations from static parameters that allow denseness and give some tractability results.



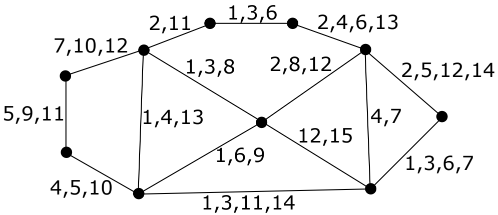
We consider parameterisations that capture properties of the input (e.g. graph structure) or desired solution (e.g. solution size).

Aim is algorithms for NP-hard problems whose running time is bounded by

$$f(k) \cdot n^c,$$

- instances of total size n
- with parameter value k
- f is any (computable) function and
- c is a fixed constant that does not depend on k

Parameterised problems admitting such an algorithm belong to the class FPT.



A temporal graph is a pair (G, λ) where G is a graph and λ maps edges of G to non-empty subsets of $\mathbb{N}$.

Given $e \in E(G)$ and $t \in \lambda(e)$, we call (e, t) a time-edge or edge appearance.

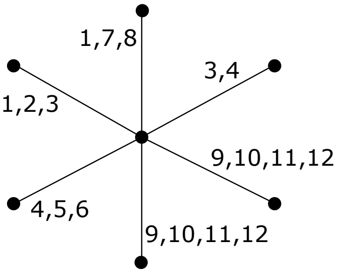
The lifetime is latest time at which any edge appears.

A (strict) temporal walk requires increasing times

STAREXP

Input: A temporal graph (S_n, λ), where S_n is a star with n leaves.

Question: Does there exist a temporal walk, starting and ending at the centre of the star, that visits every vertex?



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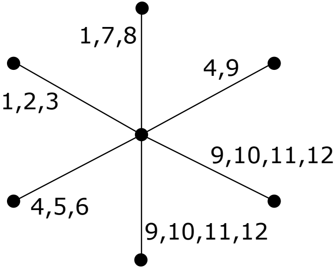

SO WE NEED PARAMETERISED APPROACHES!

How might we parameterise?

- Restrict the underlying graph
- Restrict the temporal structure
- Restrict something else: e.g. solution size

Restricting the underlying structure

- Maximum Temporal Matching is NP-hard even when G is a path (Mertzios, Molter, Niedermeier, Zamaraev & Zschoche, 2020).
- TempEuler is NP-complete even if:
  - each edge appears at only 2 times (Marino & Silva, 2021);
  - each edge appears at only 3 times, and G has feedback vertex number one (Bumpus & Meeks, 2021);
  - each edge appears at only 4 times, and G has vertex cover number 2 (Bumpus & Meeks, 2021).
- StarExp is solvable in polynomial time if each edge appears at most 3 times (Akrida, Mertzios & Spirakis, 2019), but NP-complete if edges are allowed to appear 4 or more times (Bumpus & Meeks, 2021); G is always a tree of vertex-cover number 1.

WE NEED TO CONSIDER TEMPORAL STRUCTURE

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Parameters combining times and graph structure:

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- temporal feedback edge/connection number

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Obvious temporal parameters:

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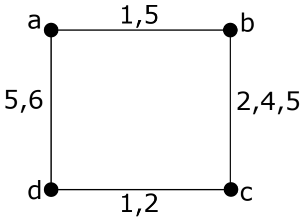
Parameters combining times and graph structure:

- timed feedback vertex number
- temporal feedback edge/connection number
- several different temporal interpretations of treewidth

The (edge) interval membership sequence of a temporal graph (G, λ) with lifetime Λ is the sequence $(F_t)_{t \in [\Lambda]}$ of edge-subsets of G where $F_t := \{e \in E(G) : \min \lambda(e) \leq t \leq \max \lambda(e)\}$.



F_1	F_2	F_3	F_4	F_5	F_6
ab	ab	ab	ab	ab	
	bc	bc	bc	bc	
cd	cd				
				ad	ad

The (edge)-interval-membership-width of (G, λ) is the integer $\mathrm{imw}(G, \lambda) := \max_{t \in [\Lambda]} |F_t|$.

Theorem (Bumpus & Meeks, 2021)

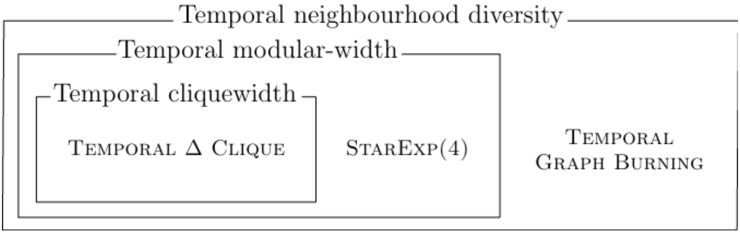
StarExp can be solved in time $\mathcal{O}(w^3 2^{3w} \Lambda)$ where Λ and w are respectively the lifetime and edge-interval-membership-width of the input graph (G, λ).

And gives progress on quite a few other problems, particularly if you look at its more-powerful cousin vertex-interval-membership-width.

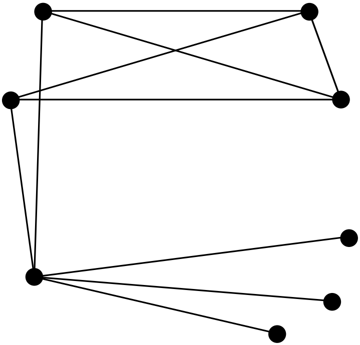
- including Temporal Graph Burning

But these require sparseness of activity, and we want to be able to deal with graphs where a lot of edges can be active at the same time.

We have started using a few generalisations from static parameters that allow lots of active edges at a time and give some tractability results.



The neighbourhood diversity of a graph $G = (V, E)$ is the smallest integer k such that V can be partitioned into sets $V_1, \ldots, V_k$ with the property that, if $x, y \in V_i$ for any i then $N(x) \setminus \{y\} = N(y) \setminus \{x\}$.



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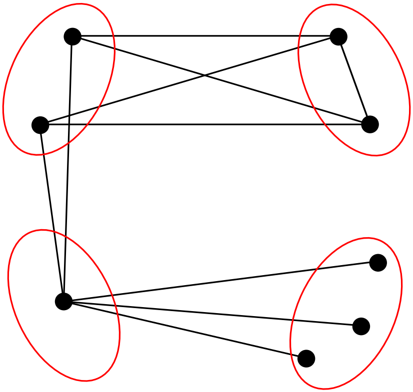
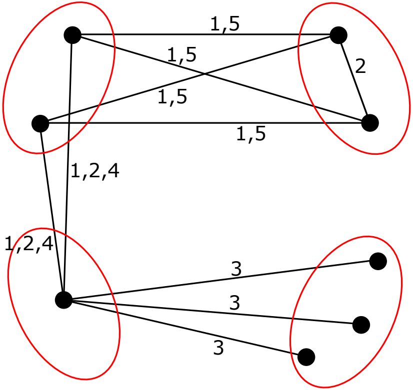


The temporal neighbourhood diversity of a temporal graph $(G = (V, E), \lambda)$ is the smallest integer k such that V can be partitioned into sets $V_1, \ldots, V_k$ with the property that, if $x, y \in V_i$ for any i then, for all times t and all vertices $z \notin \{x, y\}$, $t \in \lambda(xz)$ if and only if $t \in \lambda(yz)$.



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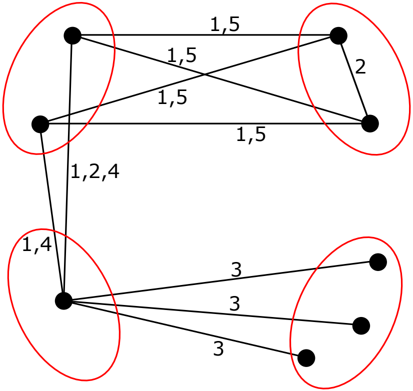

SingMinReachDelete

Input: A temporal graph (G, λ), a vertex $s \in V(G)$ and positive integers k and h.

Question: Is it possible to delete at most k time-edges from (G, λ) so that no more than h vertices are reachable from S?

Theorem

SingMinReachDelete is in FPT parameterised simultaneously by the temporal neighbourhood diversity of the input graph and the maximum number of appearances of any edge.

Temporal Graph Burning:

1. At time $t = 0$ a fire is placed at a chosen vertex. All other vertices are unburnt.
2. At all times $t \geq 1$, the fire spreads, burning all vertices u adjacent to an already burning vertex v where the edge between u and v is active at time t. Then, another fire is placed at a chosen vertex.
3. This process ends once all vertices are burning.

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Question: Does there exist a successful burning strategy for (G, λ) of length less than or equal to ℓ?

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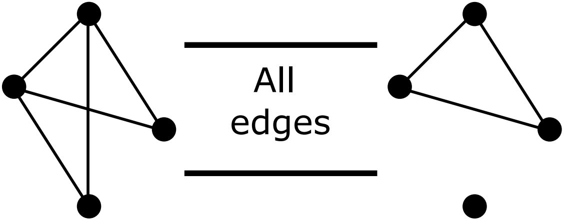
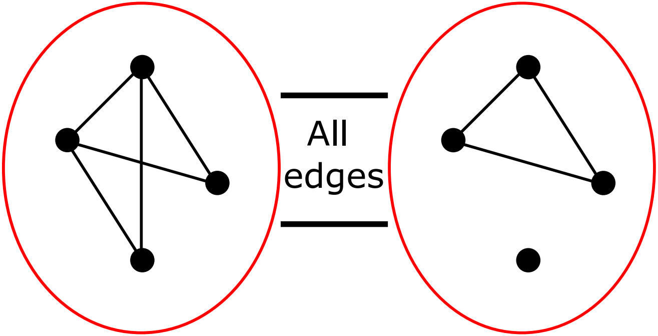
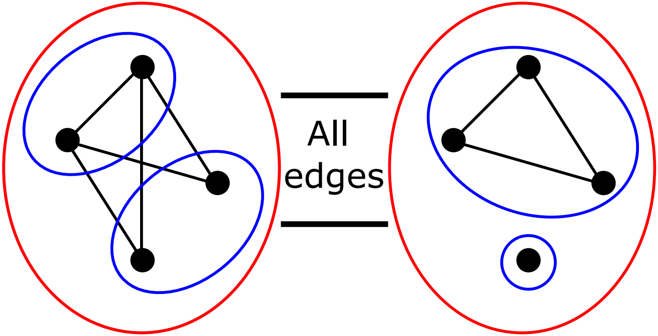
Theorem
TEMPORAL GRAPH BURNING admits an FPT algorithm parameterised by the temporal neighbourhood diversity of the input graph.


All
edges

GENERALISATION I: TEMPORAL MODULAR WIDTH

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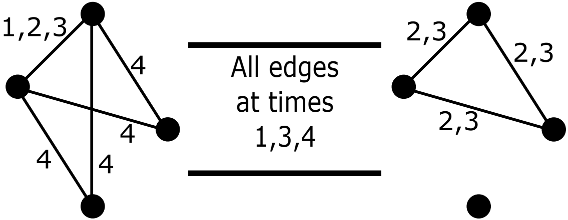
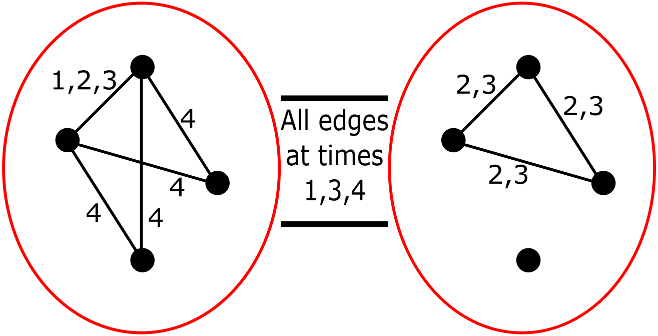

1,2,3
4
4
4
4
All edges
at times
1,3,4
2,3
2,3
2,3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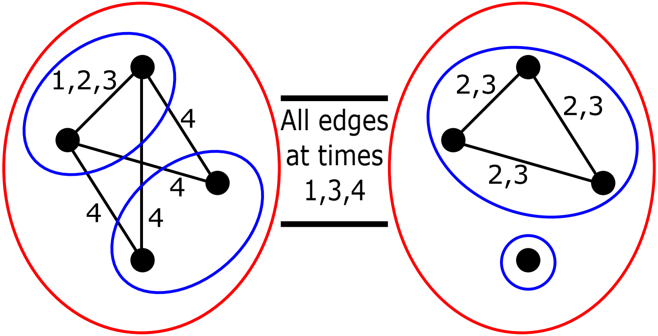
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Temporal Graph Burning is NP-hard even when restricted to graphs with constant temporal modular-width.

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Theorem

StarExp is solvable in time $(k\tau)!(k\tau)^{O(1)}$ when the temporal modular-width of the graph is at most k and every edge appears at most τ times.

GENERALISATION II: TEMPORAL CLIQUEWIDTH

- Cliquewidth is a further generalisation of modular width, which allows an additional “relabelling” operation
- Graphs of bounded modular width cannot contain long induced paths, whereas an n-vertex path has cliquewidth 3 for arbitrarily large n

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- We define a natural temporal analogue as for the previous parameters

Theorem

StarExp is NP-hard on temporal graphs with temporal cliquewidth 3.

TEMPORAL Δ CLIQUE
Input: A temporal graph $\mathcal{G} = (V, E, \lambda)$ and two integers Δ and h.
Question: Is there a set $V' \subseteq V$ of at least r vertices such that, for every $u, v \in V'$ and every window of Δ consecutive timesteps, the edge uv appears at least once in the window?

Theorem
TEMPORAL Δ CLIQUE is in FPT parameterised by the temporal cliquewidth of the input graph (provided that we are given a temporal cliquewidth construction of the input graph).

Conjecture

Any temporal graph property expressible in first-order logic admits an FPT algorithm parameterised simultaneously by the temporal cliquewidth and the length of the formula.

THE EXECUTIVE SUMMARY - AGAIN!

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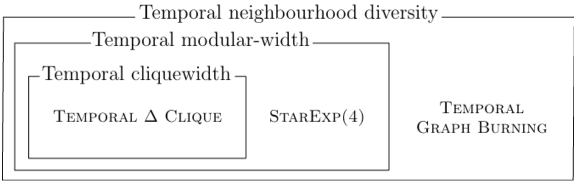

- Find more problems that are tractable parameterised by these parameters
- Is there a Courcelle-style metatheorem for temporal cliquewidth?
- Investigate the values of these parameters on real-world temporal networks

Figure 1: arxiv.org/abs/2404.19453