In Which Graph Structures Can We Efficiently Find Temporally Disjoint Paths and Walks?

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Based on a publication in IJCAI'23

Parameterized Complexity

Parameterized problem.

Each instance is associated with a parameter *k*.

Fixed-Parameter Tractability

$$k << n$$

$$f(k) \cdot n^{O(1)} = O^*(f(k))$$

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Unfortunately, not all problems are FPT!

Parameterized Analysis and Temporal Graphs.

- Nowadays, there is a very large number of works on this topic.
- Parameters: Solution size, structural parameters, **new parameters** (e.g., maximum duration).
- Classification (FPT, XP & W[1]-hard, para-NP-hard), Optimality under (S)ETH, Kernelization. Even: Counting, FPT-approximation, …

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Check Hendrik Molter's works.

- **Survey:** ``As Time Goes By: Reflections on Treewidth for Temporal Graphs'' in ``Treewidth, Kernels and Algorithms'', 2020.

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Open Problems

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The Disjoint Paths Problem. Given a graph G and a multiset of k terminal pairs $\{(s_i,t_i)\colon i\in[k]\}$, does there exist a collection of pairwise disjoint paths $\{P_i\colon i\in[k]\}$ where for every $i\in[k]$, the endpoints of P_i are s_i and t_i?

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The Graph Minors Project (23 papers, 1983-2004). Any minor-closed family of graphs can be characterized by a finite set of forbidden minors.

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Disjoint Paths Is a Central Component of the Graph Minors Project.

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Known Results on General Graphs.

Galactic FPT Algorithms: $f(k)n^3$ (RS, ’95), $f(k)n^2$ (KKR, ’12).

No Polynomial Kernel: (FHW, ‘80).

On Directed Graphs: NP-hard even for k=2 (BTY, ‘11).

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Extensively Studied on Special Graph Classes. E.g., planar graphs, split graphs, chordal graphs, …

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Temporal Paths/Walks. A temporal walk in a temporal graph is a sequence $P=((v_0,v_1,t_1),(v_1,v_2,t_2),\ldots,(v_{r-1},v_r,t_r))$, $t_1<t_2<\ldots<t_r$, such that each (v_{i-1},v_i) is an edge in the temporal graph at time t_i. When no vertex is repeated, it is a temporal path.

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The Temporally Disjoint Paths (Walks) Problem. Defined similarly to the Disjoint Paths problem.

Background

Known Results. The problems were introduced by Klobas, Mertzios, Molter, Niedermeir and Zschoche in IJCAI’21. They proved:

- **For Temporally Disjoint Paths:**
  - NP-hard when k=2,
  - NP-hard on paths,
  - FPT on trees wrt k.
- **For Temporally Disjoint Walks:**
  - W[1]-hard wrt k,
  - XP wrt k.

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But what if *k* is part of the parameterization? (Possibly, *k*>>*n* since we are given a multiset of terminal pairs.)

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Recall: KMMNZ'21 proved that TDP is FPT wrt *k* on trees.

II. Temporally Disjoint Paths is W[1]-hard wrt *k+vcn*.

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Then, use an algorithm similar to that for Temporally Disjoint Paths on trees.

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How to Prove This? Show that a walk changes direction because:

- It ``surrounds'' a solution walk that has just started or finished.
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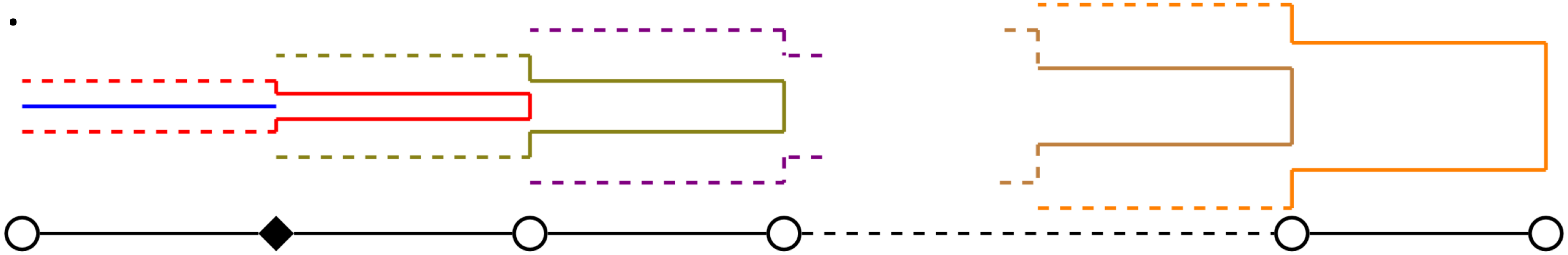
- … [k-1 times].



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2. Our hardness results hold when each of the solution's paths/walks use only constantly many edges. But, in some of them, the maximum **duration** of the solution's paths/walks is large. So: Consider the duration as part of the parameterization.
E.g.: Classify Temporally Disjoint Paths (Walks) wrt k + maximum duration.