Disentangling the Computational Complexity of Network Untangling

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Problem definitions

Given temporal graph $\mathcal{G} = (V, E_1, \ldots, E_\tau)$, a *k-activity timeline* is a set $\mathcal{T} \subseteq V \times [\tau] \times [\tau]$ such that:

- $a \leq b$ for all $(v, a, b) \in \mathcal{T}$,
- $|\{(v, a, b) \in \mathcal{T}\}| \leq k$ for all $v \in V$, and
- for all $\{u, v\} \in E_i$: there exist $a \leq i \leq b$ with $(u, a, b) \in \mathcal{T}$ or $(v, a, b) \in \mathcal{T}$.

MinTimeline$_\infty$ / MinTimeline$_+$

Input: A temporal graph $\mathcal{G} = (V, E_1, \ldots, E_\tau)$ and $k, \ell \in \mathbb{N}$.

Question: Is there a k-activity timeline $\mathcal{T}$ for $\mathcal{G}$ such that

$$\max_{(v,a,b)\in\mathcal{T}} (b-a) \leq \ell \; / \; \sum_{(v,a,b)\in\mathcal{T}} (b-a) \leq \ell?$$

Our results

For which parameters (parameter combinations) are $\textsc{MinTimeline}_\infty$ and $\textsc{MinTimeline}_+$ fixed-parameter tractable (can be solved in time $f(\mathrm{param}) \cdot |\mathcal{G}|^{\mathcal{O}(1)}$)?
Natural parameters: $|V|, k, \ell, \tau$.

Parameter	$\textsc{MinTimeline}_\infty$	$\textsc{MinTimeline}_+$
$\tau = 3$, $k = 2$, $\ell = 0$	NP-h.	NP-h.
$\lvert V\rvert + k$	FPT	FPT
$\lvert V\rvert + \ell$	W[1]-h., XP	**FPT**
$\lvert V\rvert$	W[1]-h. ($\ell = 1$), XP	XP, ?

FPT algorithm for $\textsc{MinTimeline}_+$ parameterized by $|V| + \ell$

General strategy:

1. Split solution into $\mathcal{T}_0$ containing intervals (v, a, b) with $a = b$ and $\mathcal{T}_{>0}$ containing all other intervals
2. Branching to "guess" $\mathcal{T}_{>0}$
3. For each possible choice of $\mathcal{T}_{>0}$, ILP to determine if this choice is feasible, apply Lenstra's algorithm

ILP for $\ell = 0$

Key observation: The temporal order of layers doesn't matter.

Given: temporal graph $\mathcal{G} = (V, E_1, \dots, E_\tau)$, $k \in \mathbb{N}$.
For $E \subseteq \binom{V}{2}$, let $a(E) := |\{t \in [\tau] \mid E_t = E\}|$,
$\mathcal{C}(E) := \{\text{all vertex covers of } (V, E)\}$.
ILP: variable $X_E^S \rightsquigarrow$ "how many times is vertex cover S used to cover edge set E"?

$$\sum_{S \in \mathcal{C}(E)} X_E^S = a(E), \text{ for all } E \subseteq \binom{V}{2},$$

$$\sum_{E \subseteq \binom{V}{2}} \sum_{\substack{S \in \mathcal{C}(E) \\ \text{s.t. } v \in S}} X_E^S \leq k_v, \quad \text{for all } v \in V,$$

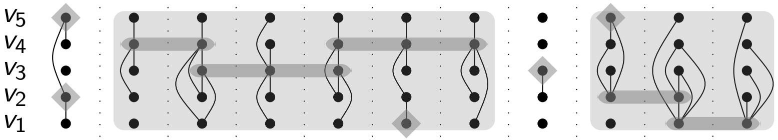
$$X_E^S \in \mathbb{N}, \quad \text{for all } E \subseteq \binom{V}{2}, S \in \mathcal{C}(E).$$

ILP for $\ell = 0$

Lenstra's algorithm **[Lenstra 1983]** $\rightsquigarrow$ FPT with respect to number of variables $O(2^{n^2})$

ILP also works for $\textsc{MinTimeline}_\infty$

Branching for $\mathcal{T}_{>0}$



- $\mathcal{T}_{>0}$ can contain at most ℓ intervals each covering at most $\ell + 1$ layers
- Consider interval graph "induced" by these intervals
- Can contain at most ℓ connected components, each covering at most $\ell + 1$ layers
- $\rightsquigarrow$ For each of the $\leq \ell$ components guess:
  1. the $\leq \ell$ graphs to be covered
  2. the intervals in the component
  3. the order of the components
- Take out the covered edges and input the rest to the algorithm $\ell = 0$

Conclusion

Parameter	MinTimeline$_\infty$	MinTimeline$_+$
$\tau = 3,\ k = 2,\ \ell = 0$	NP-h.	NP-h.
$\lvert V\rvert + k$	FPT	FPT
$\lvert V\rvert + \ell$	W[1]-h., XP	**FPT**
$\lvert V\rvert$	W[1]-h. ($\ell = 1$), XP	XP, **FPT?**