ICALP 2023
Algorithmic Aspects of Temporal Graphs

Uncertainty in temporal reachability problems

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Joint work with William Pettersson, Kitty Meeks and Jessica Enright

Motivations

Temporal graphs allow us to model many things in time sensitive networks such as transport or disease spread.

Motivations

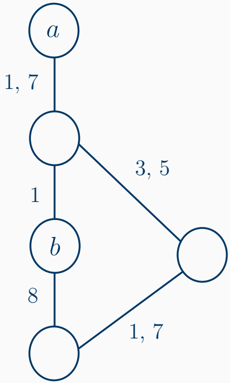
Reality may not reflect a perfect model.

Figure 1: Plot of punctuality and reliability of trains in the UK.[1]

[1] *UK Government Rail Factsheet 2022.*

Temporal Graphs

A **temporal graph**
is pair consisting of an underlying graph $G = (V, E)$ with the function $\lambda : E \to 2^{\mathbb{N}}$ that maps edges to times during which they are said to be active.



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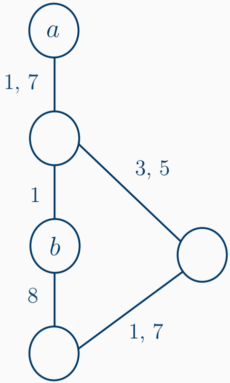

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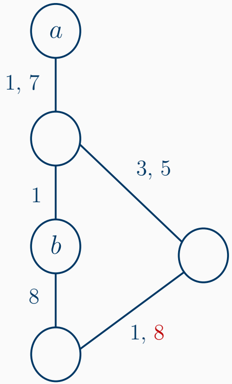

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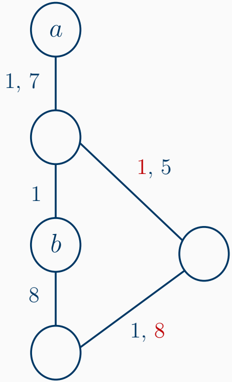
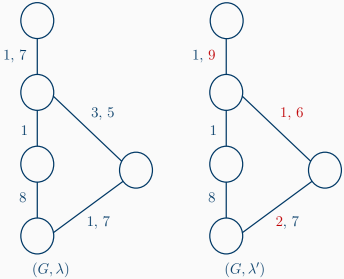


Perturbations

Given a temporal graph (G, λ), we call a temporal assignment λ' a δ-**perturbation** of λ if there is a bijection $p : (E(G), \lambda) \to (E(G), \lambda')$, $p((e, t)) = (e, t')$ where $t' \in [t - \delta, t + \delta]$ for all t.



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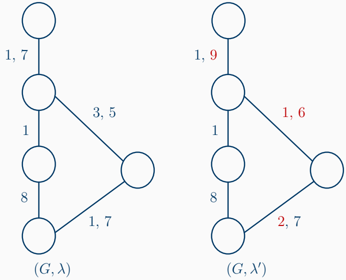
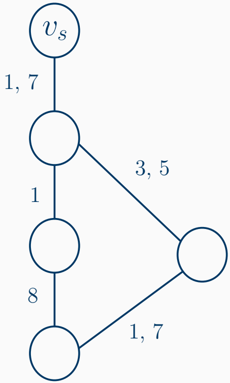
We call a perturbation λ' of a temporal assignment λ a (δ, ζ)-**perturbation** if it is a δ-perturbation and the number of changed time-edges is at most ζ.

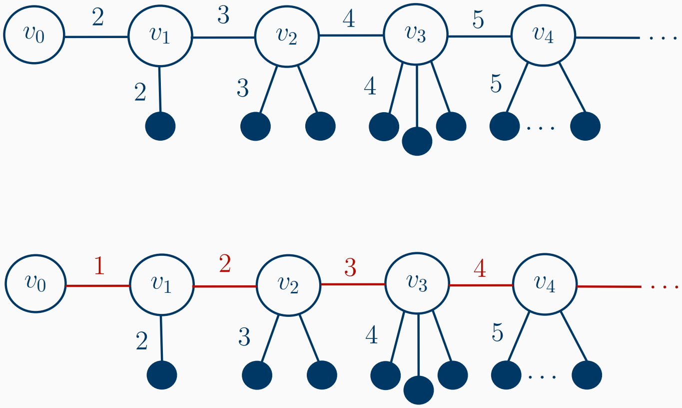


Temporal Reachability

The **reachability set** of a vertex v_s is the set of all vertices reachable from v_s by (strict) temporal path.



Uncertainty



Changing the temporal assignment by a little can cause unbounded increase in temporal reachability.

Question

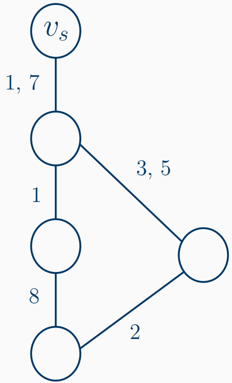
Temporal Reachability with Limited Perturbation (TRLP)
Input: A temporal graph (G, λ), a vertex v_s and positive integers ζ, k, and δ.
Question: Is there a (δ, ζ)-perturbation (G, λ') of (G, λ) such that temporal reachability of v_s in $(G, \lambda') \geq k$?

Without Uncertainty

If there is no uncertainty in the input (i.e. $\delta = \zeta = 0$), the problem is solvable in polynomial time.

If ζ is large

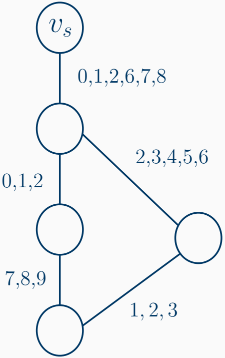
If our budget for perturbations
is at least $k - 1$, then we can ignore it.



If ζ is large

If our budget for perturbations is at least $k-1$, then we can ignore it.

We do this by relabelling each edge e with the times $[\lambda(e)-\delta, \lambda(e)+\delta]$ and calculating the maximum temporal reachability of the new temporal graph.



$\delta = 1, \zeta \geq k-1$

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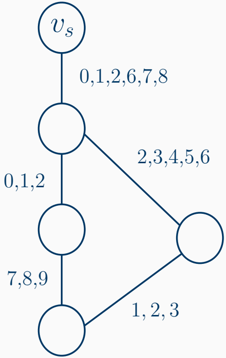
This can be done in polynomial time.



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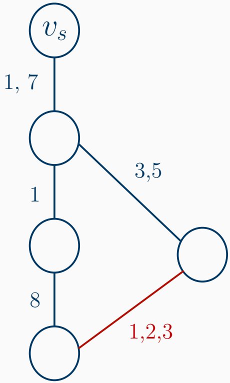
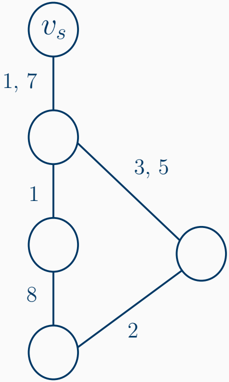
Known Perturbations

If we know
which time-edges are perturbed, we
can solve the problems in polynomial
time by just relabelling those edges.



When both δ and ζ are large

When
$\zeta \geq k - 1$ and δ is at least the lifetime of the temporal graph, we can reorder the appearances of the edges in G.



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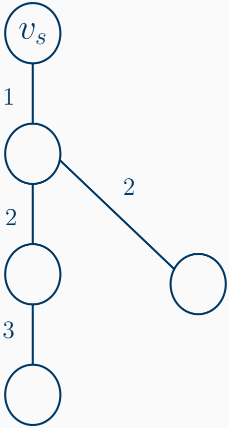

Unknown Perturbations

If $\zeta < k - 1$ and we do not know which edges are perturbed TRLP is NP-hard, even if $\delta = 1$.

Furthermore, the problem is $W[2]$-hard with respect to ζ.

Set Cover

Set Cover
Input: A universe $\mathcal{U} = \{u_1, \ldots, u_z\}$, a set $\mathcal{S} = \{S_1, \ldots, S_y\}$ of subsets of $\mathcal{U}$, and an integer c.
Question: Is there a subset $C \subseteq \mathcal{S}$ satisfying $|C| = c$ and $\bigcup_{S_i \in C} S_i = \mathcal{U}$?

TRLP Reduction Gadget

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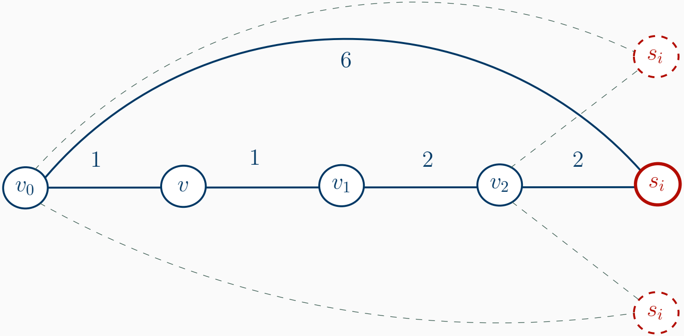

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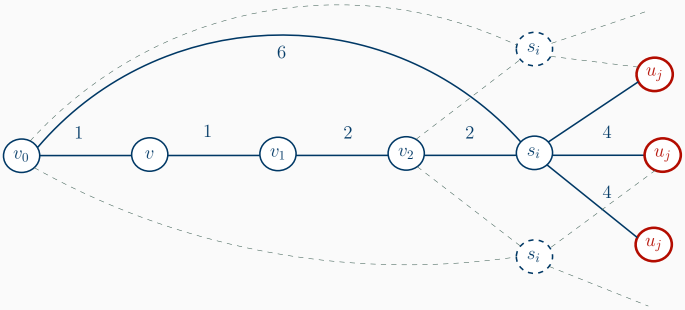

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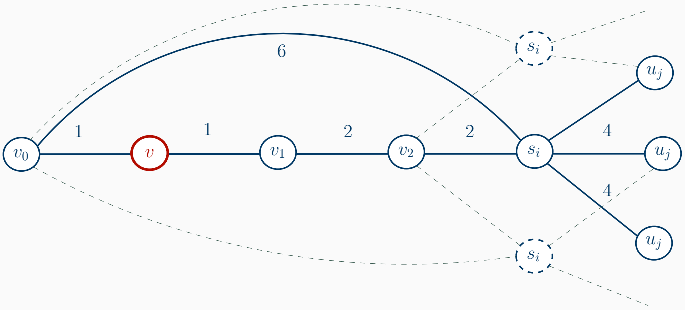

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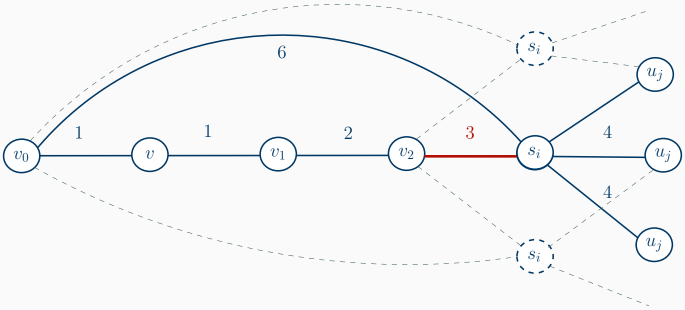
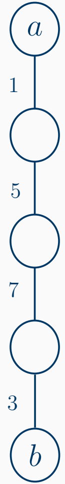


Creating a temporal path using perturbations

Given a path in a temporal graph and a time t, we can determine how many δ-perturbations are needed to make it a strict temporal path that arrives by t in polynomial time.

This is done by a dynamic program which works from the end vertex back to the beginning.



$t = 6, \delta = 4$

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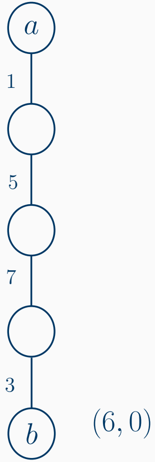
At each vertex, keep track of the time by which we need to arrive and how many perturbations we have made.



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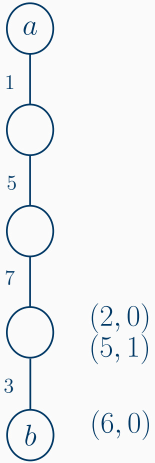

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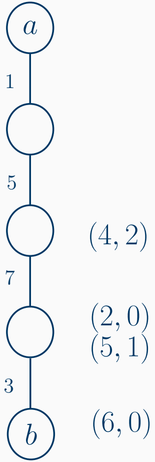

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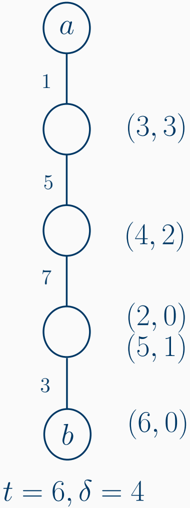

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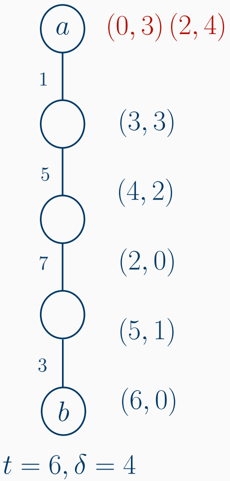

TRLP on Trees

We show that TRLP is solvable in polynomial time when the underlying graph G is a tree.

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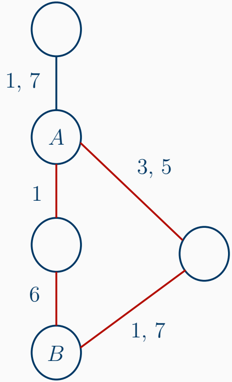
This is done by a dynamic program which works from the leaves to the root of the tree. A state consists of

1. the number of perturbations made below;
2. the maximum of vertices reachable below that vertex given this many perturbations;
3. the time of departure from that vertex;
4. the number of perturbations needed to get from the root to that vertex at that time.

Shortest, Foremost, Fastest

The **shortest**
temporal path between two vertices
is the path consisting of fewest edges.



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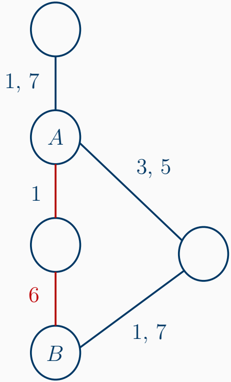

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The **fastest**
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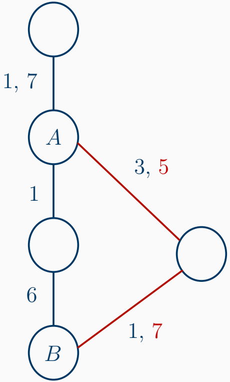
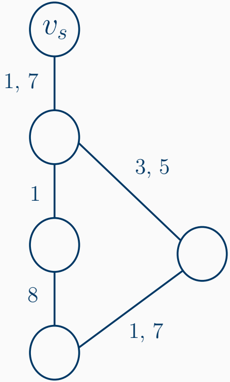
is the path with the smallest difference
between departure and arrival times.



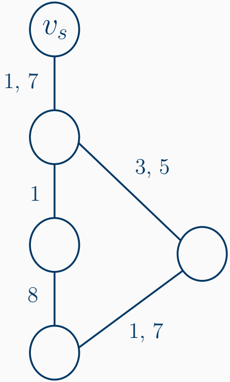
Temporal Eccentricity

v_s has **temporal shortest eccentricity** k if the shortest temporal path from v_s to every other vertex in (G, λ) has length at most k.



Temporal Eccentricity

v_s has **temporal foremost eccentricity** k if the foremost temporal path from v_s to every other vertex in (G, λ) arrives by time k.



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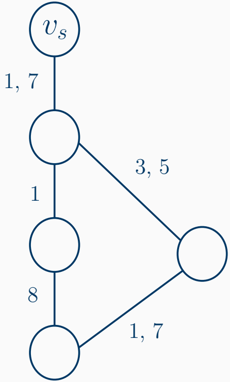
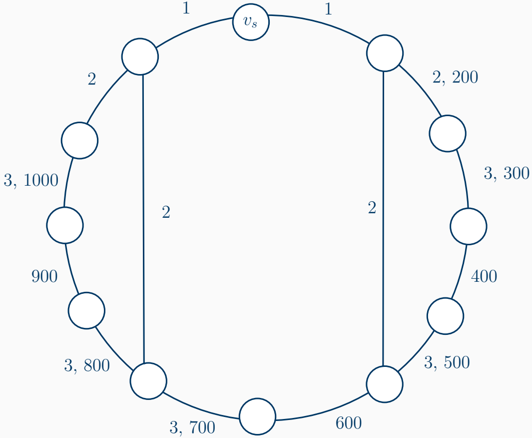


Uncertainty

We can perturb a small number of time-edges to cause a large increase in temporal eccentricity.



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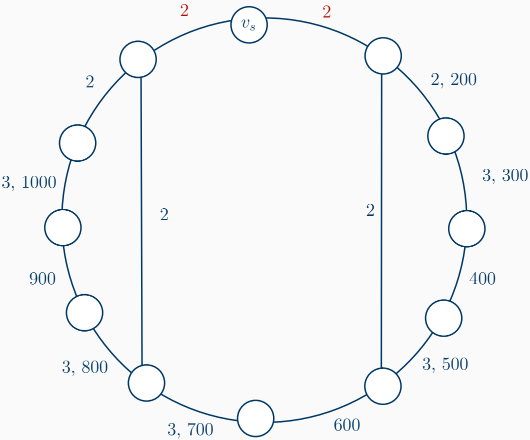

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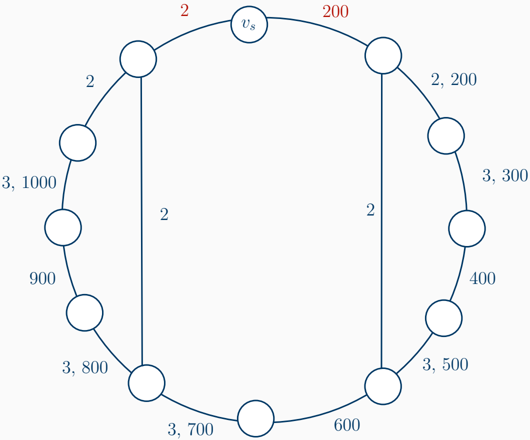

Question

Temporal ($*$) Eccentricity under Perturbation (TEP)
Input: A temporal graph (G, λ), a source vertex $v_s \in V(G)$, and positive integers ζ, k, and δ.
Question: Is there a (δ, ζ)-perturbation (G, λ') of (G, λ) such that temporal ($*$) eccentricity of v_s in (G, λ') is k?

TEP Results

	Shortest	Foremost	Fastest
No uncertainty[2]	poly(n)	poly(n)	poly(n)
Large ζ	NP-complete	poly(n)	NP-complete
Large ζ and δ	poly(n)	poly(n)	poly(n)
No restrictions	$W[2]$-hard	$W[2]$-hard	$W[2]$-hard

[2] Xuan, Ferreira, and Jarry, “Computing shortest, fastest, and foremost journeys in dynamic networks”.

Ongoing Work

This work was motivated by work on modification problems on temporal graphs. A natural question is “Can we modify a temporal graph to make it robust to (δ, ζ)-perturbations?”

There are many other problems on temporal graphs which uncertainty could be applied to.

Thanks for listening! Any questions?