Covering Timeline in Temporal Graphs: Complexity and Algorithms[1]

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July 10, 2023

[1]The talk is based on joint works with Alexandru Popa (University of Bucharest) and Manuel Lafond (Université de Sherbrooke)

Overview

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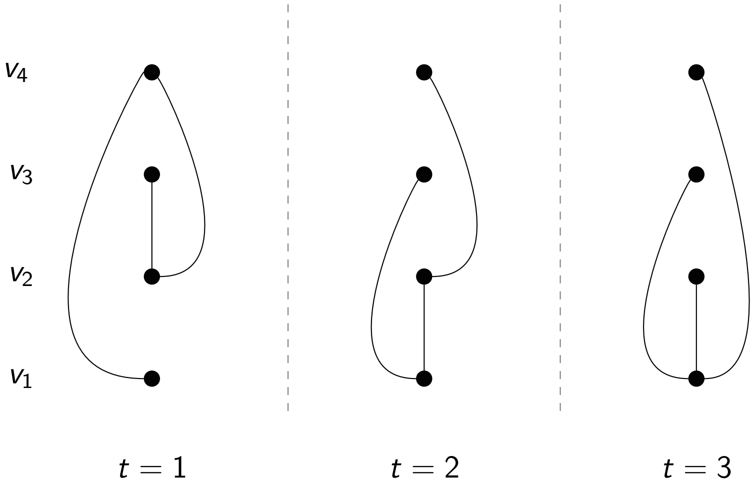
Temporal Graphs



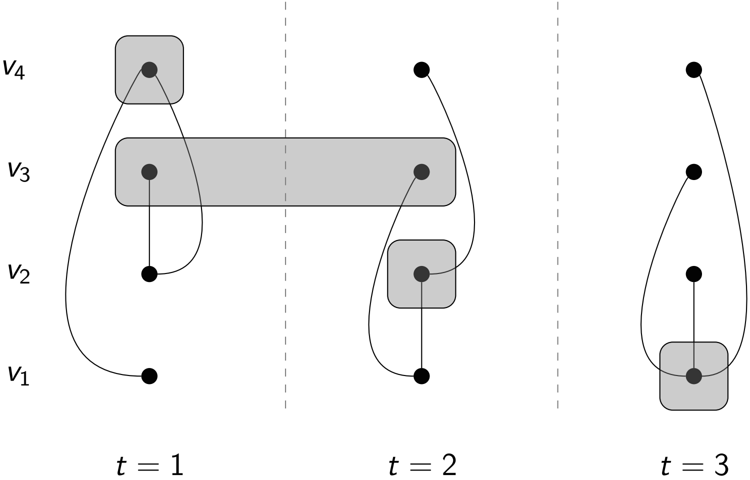
Vertex Interval Activity

For each vertex we have to define:

- **Interval activity**: time interval where the vertex is considered active (it covers edges)
- **Span** of a vertex: length - 1 of its interval activity

An **activity timeline**: a set of interval activities (one for each vertex) that **covers** edges of the temporal graph

Interval Activity



Problem Definition

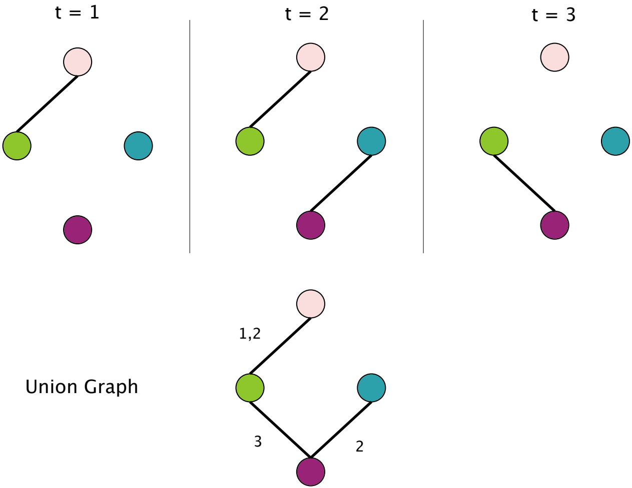
Problem (MinTCover)

Input: *A temporal graph* $G = (V, E, \mathcal{T})$
Output: *An activity timeline of minimum span that covers G*

Problem (1-MinTCover)

At most one temporal edge is present in each timestamp

Union Graph



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Previous Work

Hardness results:

- MinTCover (Rozenshtein, Tatti, Gionis 2021)
- MinTCover on two timestamps (Froese, Kunz, Zschoche 2022)
- MinTCover on three timestamps with bounded degree and 1-MinTCover (Dondi 2022)

We can compute in **polynomial time** whether there exists a solution of MinTCover that has **span equal to** 0 (Rozenshtein, Tatti, Gionis 2021)

Other Related Work

Parameterized complexity (Froese, Kunz, Zschoche 2022)

- MinTCover on two timestamps FPT parameterized by the span (via a reduction to Almost 2-SAT)
- Different other parameters (even for other variants)

Approximation (Froese, Kunz, Zschoche 2022)

- MinTCover on two timestamps cannot be approximated within a constant factor
- $O(\sqrt{\log n})$ factor for MinTCover on two timestamps

Other Related Work

Other variants of the problem

- More than one interval for vertex activity (Rozenshtein, Tatti, Gionis 2021) and (Froese, Kunz, Zschoche 2022)
- Different objective function: minimizing maximum span (Rozenshtein, Tatti, Gionis 2021)

Other temporal variants of vertex cover

- (Akrida, Mertzios, Spirakis, Zamaraev 2020), (Hamm, Klobas, Mertzios, Spirakis 2022)

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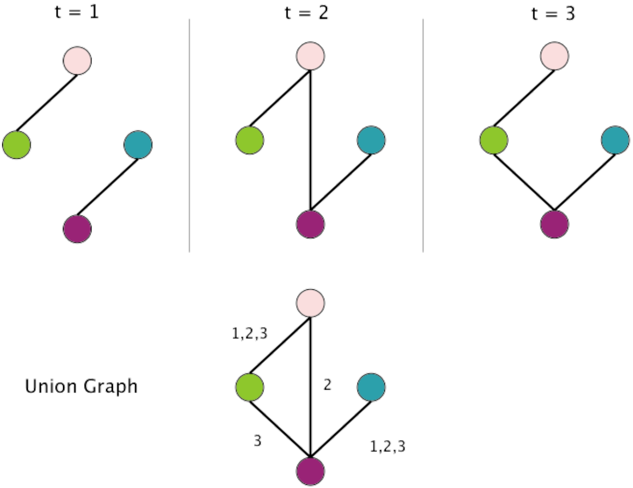
Approximation Algorithm

Combination of two approximation algorithms for the following cases:

1. Edges that occur **at least 3 times**
   - Select a vertex cover in the union graph

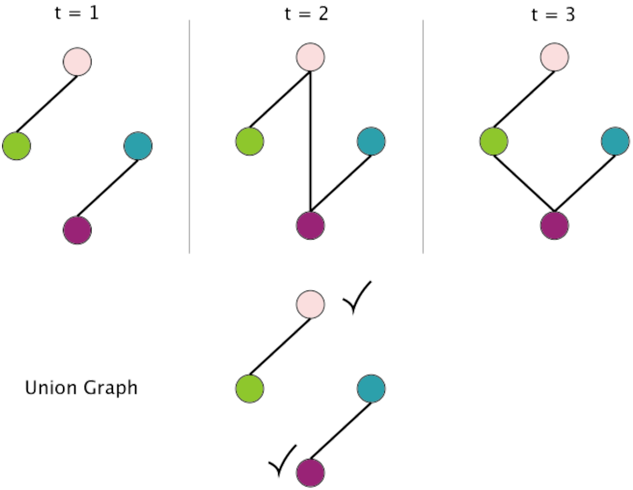
2. Edges that occur **at most 2 times**
   - Main technique: randomized rounding of an ILP formulation

Edges Occur at Least 3 Times



Union Graph

$2T$ Approximation: Edges Occur **at Least 3 Times**



Edges that Occur **at Most 2 Times**

$$\text{minimize} \sum_{v \in V} \left(\sum_{t=1}^{T} x_v^t - 1 \right) \tag{1}$$

$$\text{subject to} \sum_{t=1}^{T} x_v^t \geq 1 \qquad \forall v \in V \tag{2}$$

$$x_v^t + x_u^t \geq 1 \qquad \forall \{u, v, t\} \in E \tag{3}$$

$$x_v^t \in \{0, 1\} \qquad \forall v \in V, \ \forall t \in \{1, 2, \dots, T\} \tag{4}$$

Figure: ILP formulation for the timeline cover problem for a variant called MIN-NC-TCOVER problem

$O(T \log n)$ Approximation Algorithm: Edges Occur **at Most 2 Times**

1. Solve the LP relaxation
2. Assign 1 to a boolean variable X_v^t with probability x_v^t
3. For every vertex v such that there exist at least two variables of value 1, let t_{min} be the smallest t such that $X_v^t = 1$ and t_{max} be the maximum t such that $X_v^t = 1$. We make the vertex v active in interval $[t_{min}, t_{max}]$

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1-MinTCover: Properties

Lemma

G : instance of span k

$\mathcal{D} \subseteq V$ vertices of degree > 2

1. $|\mathcal{D}| \leq 2k$
2. *Let $G' = G \backslash \mathcal{D}$. Then, the union graph G'_U consists of a set of disjoint paths*

Lemma

Optimal 1-MinTCover on G has a span of $k \iff$ feedback vertex set of G_U is at most $2k$

$4(T-1)$-approximation algorithm for 1-MinTCover

1. Compute a 2-approximate feedback vertex set F of G_U (Bafna, Berman, Fujito 1999)
2. Make each vertex $v \in F$ active in the time interval $[1, T]$
3. $G - F$ is acyclic

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FPT Algorithm for MinTCover

FPT algorithm for MinTCover for parameter the span of the solution

The algorithm consists of two parts:

1. **Iterative compression**
2. An **FPT reduction** to the Directed Pair Cut problem

Iterative Compression

High level idea of the iterative compression part:

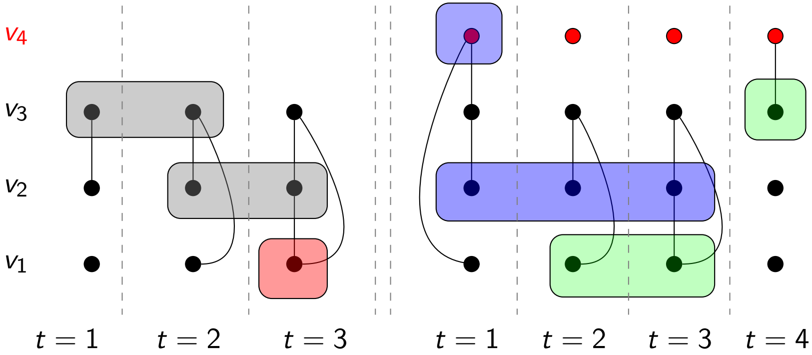
1. Consider a solution S_h of MinTCover on the the temporal subgraph induced by $\{v_1, \dots, v_h\}$
2. Compute whether there exists a solution S_{h+1} of MinTCover on the the temporal subgraph induced by $\{v_1, \dots, v_h\} \uplus \{v_{h+1}\}$

Iterative Compression

Some details on the computation of S_{h+1} from S_h:

1. Branch on the **interval activity of vertex** v_{h+1}
2. Branch on interval activity of some vertices with **positive span** in S_h: N_h
3. FPT reduction to DIRECTED PAIR CUT for the definition of timeline activity of other vertices

Iterative Compression



FPT reduction to Directed Pair Cut

Problem (Directed Pair Cut)

Input: *A directed graph D, a source vertex s and a set $\mathcal{P}$ of pairs*
Output: *A cut of at most k arcs such that for each $(u, v) \in \mathcal{P}$ at most one of u and v is reachable from s*

Some details on the reduction to Directed Pair Cut

1. For each vertex not in $N_h \uplus \{v_{h+1}\}$, construct a gadget
2. Pairs are used to force edges to be covered

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Open Problems

Open problems for MinTCover:

1. **Approximation complexity**: analyze the dependency on T of the approximation factor for MinTCover and 1-MinTCover
2. **Parameterized complexity**: Extend the technique to the case where vertex activity is defined as > 1 interval?

Thank you!