Defining Labeling that Preserves Reachability

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Algorithmic Aspects of Temporal Graphs V

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Overview

- Notation
- Problem definition
- Results

Basic Definitions

Definition

A temporal graph $\mathcal{G}$ is a pair (G, λ) where:

- $G = (V, E)$ is an underlying (di)graph and
- $\lambda : E \to 2^{\mathbb{N}}$ is a discrete time-labeling function.

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Definition

A temporal graph $\mathcal{G}$ is (temporally) connected iff for all $u, v \in V(G)$ there exists a temporal (u, v)-path[1].
Let $R \subseteq V(G)$, $\mathcal{G}$ is R-(temporally) connected iff for all $u, v \in R$ there exists a temporal (u, v)-path.

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Notations

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- The total cost of (G, λ) is $|\lambda| = \sum_{e \in E} |\lambda_e|$.
- $\kappa(G, \alpha)$ is the minimum cost of (G, λ), where the maximum label used by λ is α.

Problem(s) Definitions

Min. Labeling (ML)

Input: A static graph $G = (V, E)$ and an integer $k \in \mathbb{N}$.
Question: Does there exist a temporally connected temporal graph (G, λ), where $|\lambda| \leq k$?

Problem(s) Definitions

Min. Aged Labeling (MAL)

Input: A static graph $G = (V, E)$ and two integers $k, a \in \mathbb{N}$.
Question: Does there exist a temporally connected temporal graph (G, λ), where $|\lambda| \leq k$ and $\alpha(\lambda) \leq a$?

PROBLEM(S) DEFINITIONS

Min. Labeling (ML)

Input: A static graph $G = (V, E)$ and an integer $k \in \mathbb{N}$.
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Min. Steiner Labeling (MSL)

Input: A static graph $G = (V, E)$, a subset $R \subseteq V$ and an integer $k \in \mathbb{N}$.
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Temp. connected		
R-connected		

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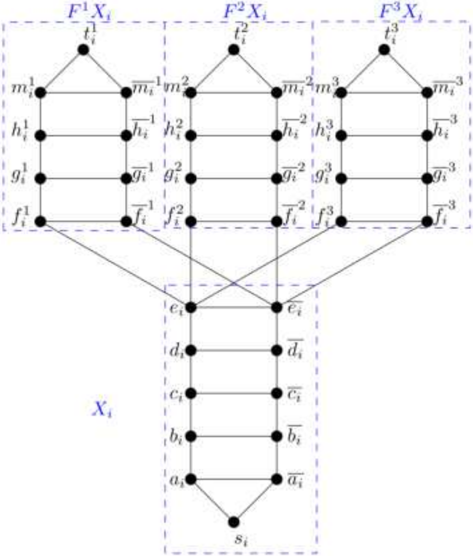
Reduction from Monotone Max XOR(3):

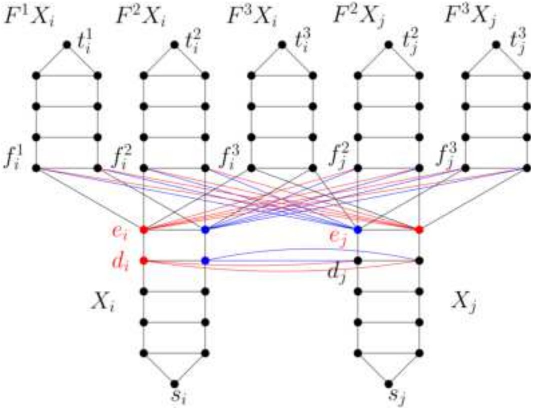
- conjunction of XOR clauses,
- non-negated variables,
- variables appear exactly 3 times.

Variable gadgets



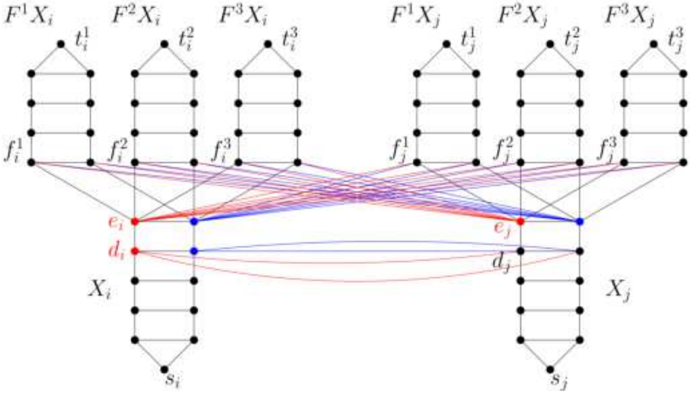
Connecting variable gadgets I

Clause $(x_i \oplus x_j)$ with 3rd and 1st appearance of x_i, x_j, respectively.



Connecting variable gadgets II

No clause with x_i and x_j.



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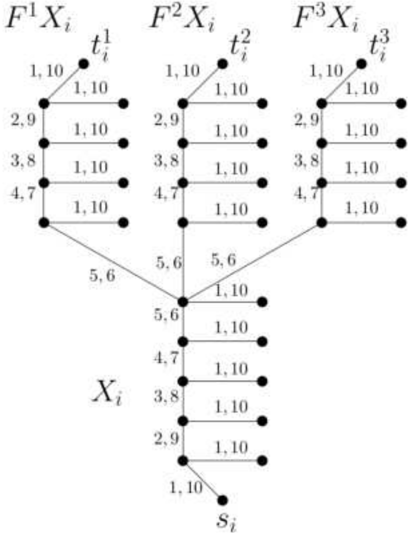

Idea of the Proof

- $\alpha = d = 10$,
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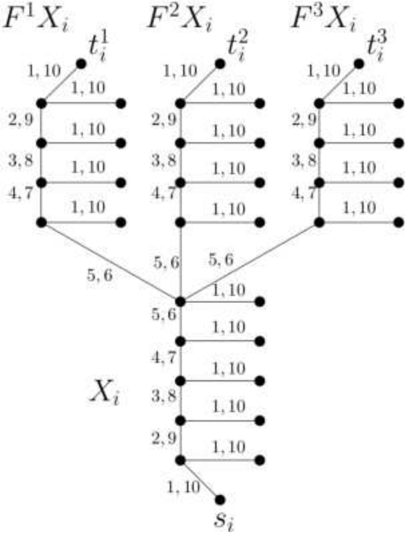
- a clause is satisfied iff only one side of the shared fork is labeled.



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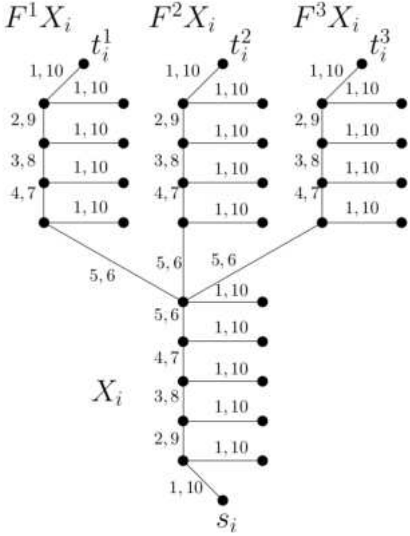
$$OPT(G_\phi, d_\phi) \leq \text{poly}(n, k)$$
$$\Leftrightarrow$$
$$OPT(\phi) \geq k.$$



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- Use an FPT algorithm for Steiner Tree.

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Crucial property:

- There exists a minimum labeling that is a tree or a tree with a C_4.

Idea of the algorithm:

- Use an FPT algorithm for Steiner Tree.
- Iterate over all C_4s in G, check if one can be labeled in an optimum solution.

Thank you!