Approximating Multistage Matching Problems
and other subgraph problems

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Reminder: Perfect Matching

**Given:** Compatibility graph.

**Task:** Find a unique compatible parrot for each pirate.

**Definition (Perfect Matching)**

A *perfect matching* in a graph $G = (V, E)$ is a set $M \subseteq E$ of edges, such that

- no two edges in $M$ share an endpoint,
- each vertex in $V$ is incident with an edge in $M$.
Motivation: Multistage Perfect Matching

**Task:** Find a perfect matching in each year such that the sum of common edges in consecutive years is maximized.
A **multistage graph** is a tuple $G = (V, E_1, \ldots, E_\tau)$ consisting of a set of vertices $V$ and multiple sets of edges $E_i \subseteq \binom{V}{2}$.

The graph induced by some $E_i$ is called the $i$-th stage of $G$ and denoted $G_i$. 

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**Formal setting**

**Definition**

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Formal setting

Definition

- A multistage perfect matching in $G$ is a sequence of matchings $\mathcal{M} = (M_i)_{i=1}^{\tau}$ such that each $M_i$ is a perfect matching in $G_i$. The profit of $\mathcal{M}$ is $p(\mathcal{M}) := \sum_{i=1}^{\tau-1} |M_i \cap M_{i+1}|$.

- MIM is the problem of finding an $\mathcal{M}$ that maximizes $p(\mathcal{M})$. 
Related work

- Deciding **MIM** is NP-hard...
  - ...for $\geq 6$ stages [Gupta et al. 2014],
  - ...for $\geq 2$ stages [Bampis et al. 2018],
  - ...for $\geq 2$ stages & each stage consists only of disjoint cycles

- **Maximum Multistage Matching** (with edge weights) is APX-hard, but there is a $1/2$-approximation [Bampis et al. 2018].
  
  **Task:** Maximize $p(M) + \sum_{i=1}^{T} w(M_i)$.

  Doesn’t this include our problem?
Multistage Perfect Matching

\[ p(\mathcal{M}) + \sum_{i=1}^{\tau} w(\mathcal{M}_i) = (1 + 1) + (3 + 3 + 3) = 11 \]

\[ p(\mathcal{M}') + \sum_{i=1}^{\tau} w(\mathcal{M}_i') = (2 + 2) + (3 + 2 + 3) = 12 \]
Approximation under hard constraints for each stage
Preficiency & Intertwinement

Perfect Matching is preficient (=preference efficient)

Given: Graph $G = (V, E)$, edge set $P \subseteq E$.
Task: Compute a perfect matching $M$ on $G$ that maximizes $|M \cap P|$.

prefPM($G, F$)

```plaintext
foreach $e \in E \setminus P$ do
    $w(e) \leftarrow 1$
foreach $e \in P$ do
    $w(e) \leftarrow 1 + \varepsilon$
compute a maximum weight matching $M$ on $G$
return $M$
```

Intertwinement: $\chi := \max_{i < \tau} |E_i \cap E_{i+1}|$

2 stages: $E_\cap := E_1 \cap E_2$, $\chi := |E_\cap|$
Approximation for two stages: “prefer-and-ignore”

2IM-Approx

\[ P \leftarrow E_n \]

\[ \text{while } |P| > 0 \text{ do} \]

\[ M_1 \leftarrow \text{prefPM}(G_1, P) \]

\[ M_2 \leftarrow \text{prefPM}(G_2, M_1) \]

\[ P \leftarrow P \setminus M_1 \]

\text{return that } (M_1, M_2) \text{ from above with maximal } p(M_1, M_2) \]
Theorem

\textbf{2IM-Approx} is a (tight) $1/\sqrt{2\chi}$-approximation.

Proof sketch.

Let $M^*_n = M^*_1 \cap M^*_2$ be optimal. Assume $p(M^*_1, M^*_2) = |M^*_n| \geq 1$.

Case 1: $|M^*_n| \leq \sqrt{2\chi}$ is “small”.
⇒ Any solution with profit $|M_1 \cap M_2| \geq 1$ suffices.

Case 2: $|M^*_n| > \sqrt{2\chi}$ is “large”.
➤ Suppose in each iteration only “few” edges in $M_1 \cap M^*_n \cap P$.
⇒ We need “many” iterations, but ...
    ... the number of remaining edges in $M^*_n \cap P$ decreases “slowly”.
⇒ $M_1$ prefers $M^*_n \cap P$.
⇒ Eventually, $M_1 \cap M^*_n \cap P$ is “large”. ⚡
➤ $M_2$ maximizes $|M_1 \cap M_2| \geq |M_1 \cap M^*_2|$.
Further results for matchings

- **MIM** is NP-hard already in extremely restricted settings
- $1/\sqrt{2\chi}$-approximation for 2IM
- $1/\sqrt{8\chi}$-approximation for MIM
- Natural ILP for 2IM has LP-gap of $\sqrt{\chi}$
- If MIM is APX-hard, so is 2IM

- Approx. algorithms for MUM (=minimize unions)

$\alpha = 1/\sqrt{2\chi}$

$\alpha(\chi) \mapsto \alpha((\tau - 1)\chi)$

$2 - \alpha$
Beyond matchings...

**Subgraph Problem (SP):** *Intuition* Given a graph, find a subset of its graph elements that optimizes some measure.

**Examples:** Matching, Shortest Path, Vertex Cover, Independent Set, Max. Planar Subgraph,…

**Multistage Subgraph Problem (MSP):** *Intuition* Given an SP, find optimal solutions for each stage. **Maximize** transition profit.

**Theorem**

Consider a **preficient** MSP where we maximize the intersection between consecutive stages.

On two stages, it allows a $1/\sqrt{2|\chi|}$-approximation.

On arbitrarily many stages, it allows a $1/\sqrt{8|\chi|}$-approximation.
Preficiency?

**Theorem**

Consider a **preficient** MSP where we maximize the intersection between consecutive stages.

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**Preficient MSP**: underlying SP allows a polynomial algorithm that prefers some graph elements over others.

Preficiency is typically **trivial** to show

(add some small $\varepsilon > 0$ to cost function)

$\implies$ Theorem applicable to, e.g., the NP-hard multistage versions of:

- **Shortest s-t-path**, **Minimum s-t-cut**, **Maximum s-t-cut** on weakly-bipartite graphs (superset of planar graphs),
- **Minimum-Weight Vertex Cover** on bipartite graphs,
- **Maximum-Weight Independent Set** on bipartite graphs,...
Core question in all our investigations:
How well can be approximate a multistage problem if we require \textbf{optimal} solutions in each stage.

Open questions:

- We always end up with approximation ratios $\Theta(1/\sqrt{\chi})$. Is this best-possible for general MSPs? For matchings?
- What about optimal transitions but suboptimal per-stage solutions?

Thank you!