Approximating Multistage Matching Problems

and other subgraph problems

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joint work with **Niklas Troost** and **Tilo Wiedera**

"Approximating Multistage Matching Problems." IWOCA 2021.

"A General Approach to Approximate Multistage Subgraph Problems."
arXiv 2107.02581.

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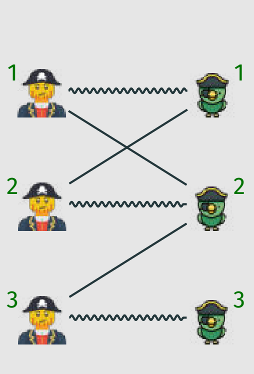
Reminder: Perfect Matching



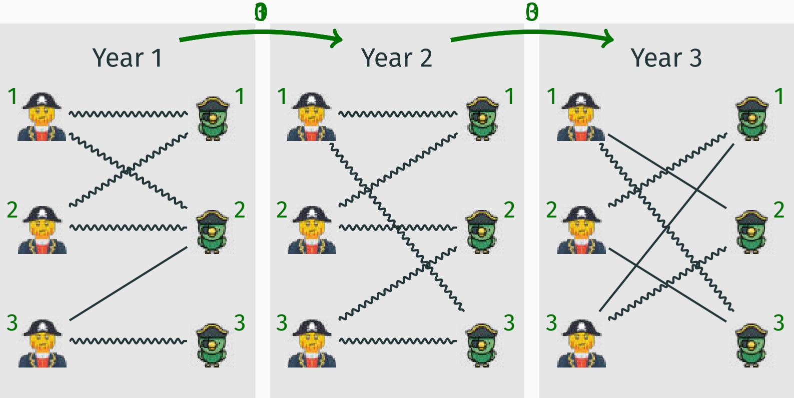
Given: Compatibility graph.

Task: Find a unique compatible parrot for each pirate.

Definition (Perfect Matching)

A *perfect matching* in a graph $G = (V, E)$ is a set $M \subseteq E$ of edges, such that

- no two edges in M share an endpoint,
- each vertex in V is incident with an edge in M.

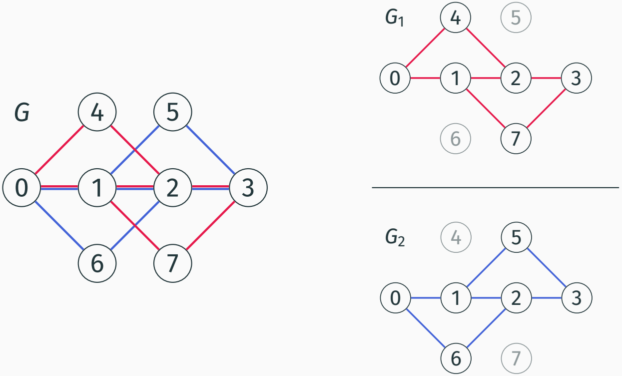


Task: Find a perfect matching in each year such that the sum of common edges in consecutive years is maximized.

Formal setting

Definition

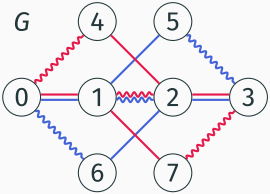
- A **multistage graph** is a tuple $G = (V, E_1, \ldots, E_\tau)$ consisting of a set of vertices V and multiple sets of edges $E_i \subseteq \binom{V}{2}$.
- The graph induced by some E_i is called the i-th **stage** of G and denoted G_i.

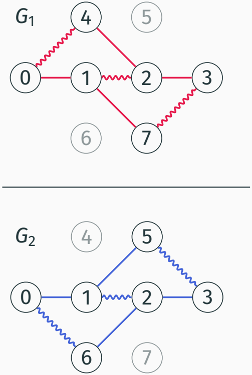


Formal setting

Definition

- A **multistage perfect matching** in G is a sequence of matchings $\mathcal{M} = (M_i)_{i=1}^{\tau}$ such that each M_i is a perfect matching in G_i. The **profit** of $\mathcal{M}$ is $p(\mathcal{M}) := \sum_{i=1}^{\tau-1} |M_i \cap M_{i+1}|$.
- **MIM** is the problem of finding an $\mathcal{M}$ that maximizes $p(\mathcal{M})$.





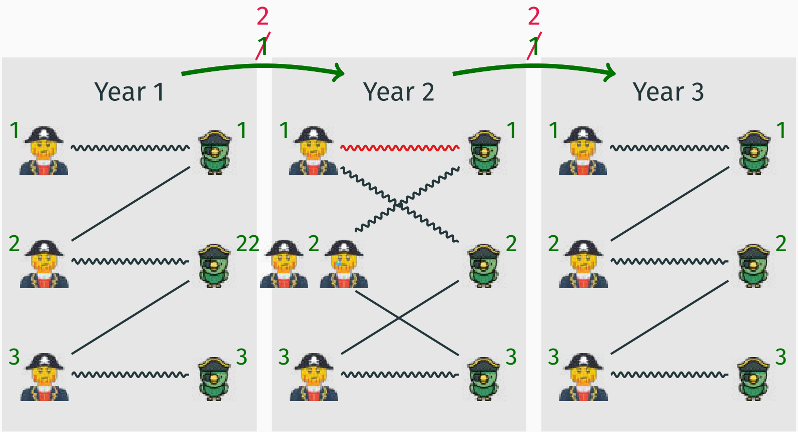
Related work

- Deciding **MIM** is NP-hard...
  ...for ≥ 6 stages [Gupta et al. 2014],
  ...for ≥ 2 stages [Bampis et al. 2018],
  ...for ≥ 2 stages & each stage consists only of disjoint cycles

- **Maximum Multistage Matching** (with edge weights) is APX-hard, but there is a $1/2$-approximation [Bampis et al. 2018].
  Task: Maximize $p(\mathcal{M}) + \sum_{i=1}^{\tau} w(M_i)$.

Doesn't this include our problem?

Multistage Perfect~~Perfect~~ Matching



$$p(\mathcal{M}) + \sum_{i=1}^{\tau} w(M_i) = (1+1) + (3+3+3) = 11$$

$$p(\mathcal{M}') + \sum_{i=1}^{\tau} w(M_i') = (2+2) + (3+2+3) = 12$$

Approximation under **hard constraints** for each stage

Preficiency & Intertwinement

Perfect Matching is preficient (=preference efficient)

Given: Graph $G = (V, E)$, edge set $P \subseteq E$.
Task: Compute a perfect matching M on G that maximizes $|M \cap P|$.

prefPM(G, F)

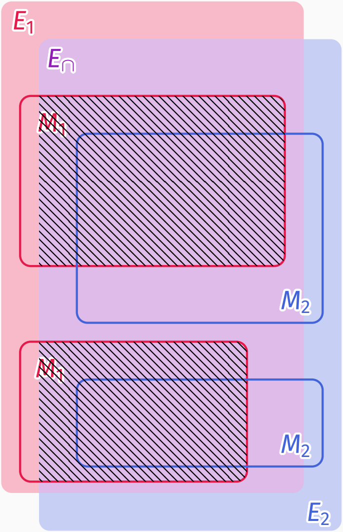
foreach $e \in E \setminus P$ **do**
    $w(e) \leftarrow 1$
foreach $e \in P$ **do**
    $w(e) \leftarrow 1 + \varepsilon$
compute a maximum weight matching M on G
return M

Intertwinement: $\chi := \max_{i<\tau} |E_i \cap E_{i+1}|$
2 stages: $E_\cap := E_1 \cap E_2$, $\chi := |E_\cap|$

Approximation for two stages: “prefer-and-ignore”

2IM-Approx

$P \leftarrow E_\cap$
while $|P| > 0$ **do**
  $M_1 \leftarrow \text{prefPM}(G_1, P)$
  $M_2 \leftarrow \text{prefPM}(G_2, M_1)$
  $P \leftarrow P \setminus M_1$
return *that* (M_1, M_2) *from above with maximal* $p(M_1, M_2)$



Approximation for two stages: Proof sketch

Theorem

2IM-Approx is a (tight) $1/\sqrt{2\chi}$-approximation.

Proof sketch.

Let $M^*_\cap = M^*_1 \cap M^*_2$ be optimal. Assume $p(M^*_1, M^*_2) = |M^*_\cap| \geq 1$.

Case 1: $|M^*_\cap| \leq \sqrt{2\chi}$ is "small".
$\Rightarrow$ Any solution with profit $|M_1 \cap M_2| \geq 1$ suffices.

Case 2: $|M^*_\cap| > \sqrt{2\chi}$ is "large".

- Suppose in each iteration only "few" edges in $M_1 \cap M^*_\cap \cap P$.

$\Rightarrow$ We need "many" iterations, but ...
... the number of remaining edges in $M^*_\cap \cap P$ decreases "slowly".

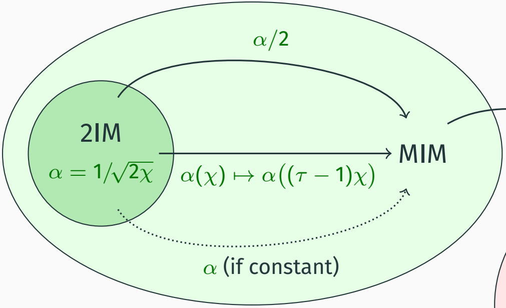
$\Rightarrow$ M_1 prefers $M^*_\cap \cap P$.

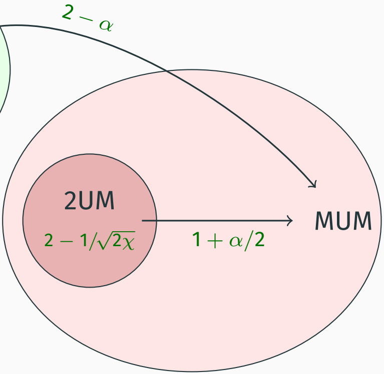
$\Rightarrow$ Eventually, $M_1 \cap M^*_\cap \cap P$ is "large". ↯

- M_2 maximizes $|M_1 \cap M_2| \geq |M_1 \cap M^*_2|$.

□

Further results for matchings





- **MIM** is NP-hard already in extremely restricted settings
- $1/\sqrt{2\chi}$-approximation for **2IM**
- $1/\sqrt{8\chi}$-approximation for **MIM**
- natural ILP for **2IM** has LP-gap of $\sqrt{\chi}$
- If **MIM** is APX-hard, so is **2IM**

- Approx. algorithms for **MUM** (=minimize unions)

Beyond matchings...

Subgraph Problem (SP): *[Intuition]* Given a graph, find a subset of its graph elements that optimizes some measure.

Examples: Matching, Shortest Path, Vertex Cover, Independent Set, Max. Planar Subgraph,...

Multistage Subgraph Problem (MSP): *[Intuition]* Given an SP, find **optimal** solutions for each stage. **Maximize** transition profit.

Theorem

*Consider a **preficient** MSP where we maximize the intersection between consecutive stages.*
On two stages, it allows a $1/\sqrt{2|\chi|}$*-approximation.*
On arbitrarily many stages, it allows a $1/\sqrt{8|\chi|}$*-approximation.*

Preficiency?

Theorem

*Consider a **preficient** MSP where we maximize the intersection between consecutive stages.*
On two stages, it allows a $1/\sqrt{2|\chi|}$-approximation.
On arbitrarily many stages, it allows a $1/\sqrt{8|\chi|}$-approximation.

Preficient MSP: underlying SP allows a polynomial algorithm that prefers some graph elements over others.

Preficiency is typically **trivial** to show

(add some small $\varepsilon > 0$ to cost function)

$\Longrightarrow$ Theorem applicable to, e.g., the NP-hard multistage versions of:

- **Shortest s-t-path**, **Minimum s-t-cut**, **Maximum s-t-cut** on weakly-bipartite graphs (superset of planar graphs), **Minimum-Weight Vertex Cover** on bipartite graphs, **Maximum-Weight Independent Set** on bipartite graphs,...

Summary

Core question in all our investigations:

How well can be approximate a multistage problem if we require **optimal** solutions in each stage.

Open questions:

- We always end up with approximation ratios $\Theta(1/\sqrt{\chi})$.
  Is this best-possible for general MSPs? For matchings?
- What about optimal transitions but suboptimal per-stage solutions?

Thank you!