# Temporal Graph Problems From the Multistage Model

Till Fluschnik

Technische Universität Berlin, Germany,
Algorithmics and Computational Complexity, Faculty IV

Workshop: Algorithmic Aspects of Temporal Graphs III

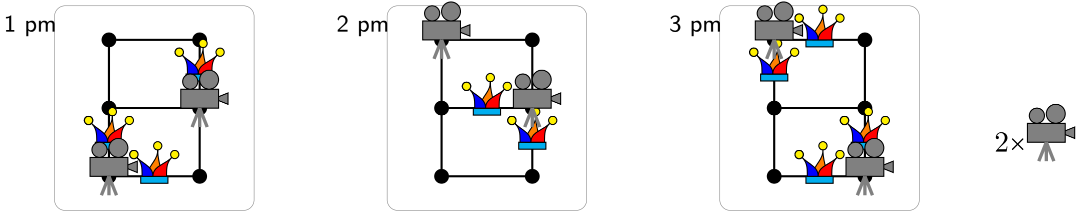
July 07, 2020

Based on joint work with Rolf Niedermeier, Valentin Rohm, Carsten Schubert, and Philipp Zschoche.

# The Multistage Model

Several instances over time (stages) of the same problem. Find a solution to each instance such that the sequence of solutions is robust, i.e., consecutive solutions differ not too much.

**Example:** (Saarbrückener Faasend) Asked to film (live) street parades (e.g. carnival); We have few camera teams; We want few concurrent resets.




Input: A sequence $(I_1, \ldots, I_\tau)$ of instances of some problem $L$ (e.g. Vertex Cover).

Question: Is there a sequence $(S_1, \ldots, S_\tau)$ of solutions, i.e., $S_j$ is a solution to $I_j$ for all $j \in \{1, \ldots, \tau\}$, such that $\operatorname{diff}(S_j, S_{j+1})$ is small for all $j \in \{1, \ldots, \tau - 1\}$?

# From Multistage To Temporal Graph Problem

A multistage graph problem:

**Input:** A sequence $(I_1 = (G_1, k), \ldots, I_\tau = (G_\tau, k))$ of instances of Vertex Cover over the **same** set $V$ of vertices, i.e. $G_i = (V, E_i)$.

**Question:** Is there a sequence $(S_1, \ldots, S_\tau)$ of solutions, i.e., $S_j$ is a solution to $I_j$ for all $j \in \{1, \ldots, \tau\}$, such that $\text{diff}(S_j, S_{j+1})$ is small for all $j \in \{1, \ldots, \tau - 1\}$?

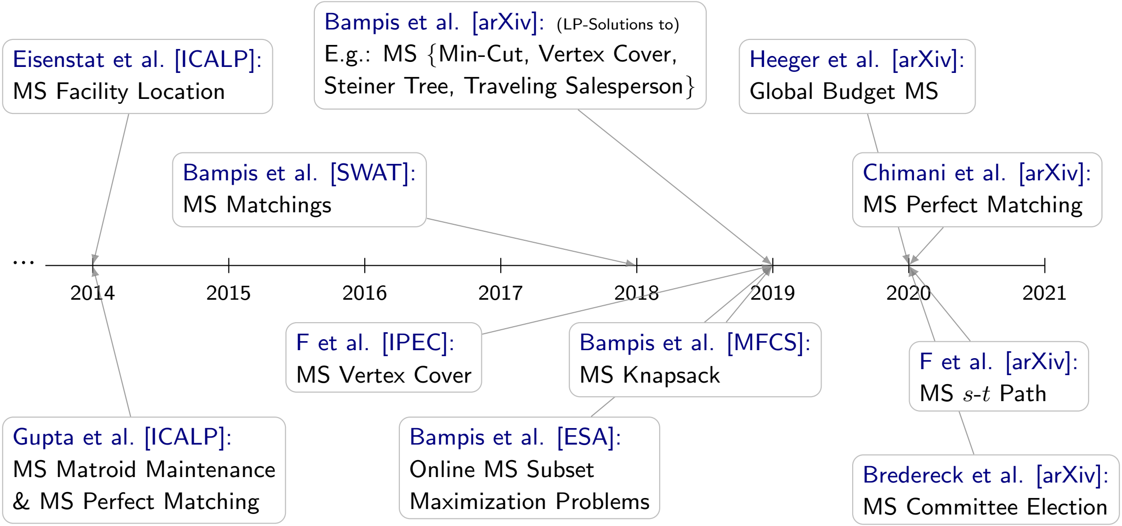
$$\Downarrow$$

**Input:** A temporal graph $\mathcal{G} = (V, E_1, \ldots, E_\tau)$, and $k \in \mathbb{N}$.

**Question:** Is there a sequence $(S_1, \ldots, S_\tau)$ such that $S_j \subseteq V$ is a size-at-most-$k$ vertex cover of $(V, E_j)$ for all $j \in \{1, \ldots, \tau\}$ and $\text{diff}(S_j, S_{j+1})$ is small for all $j \in \{1, \ldots, \tau - 1\}$?

↑ $|S_j \triangle S_{j+1}| \leq \ell$ for some given $\ell \in \mathbb{N}$

# A Brief History on “Multistage (MS)”

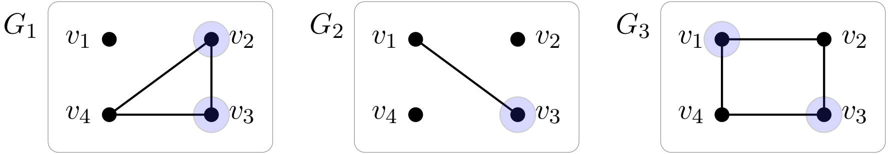


# Multistage Vertex Cover

Multistage Vertex Cover (MSVC)

**Input:** A temporal graph $\mathcal{G} = (V, E_1, \ldots, E_\tau)$, two integers $k, \ell \in \mathbb{N}$.

**Ques.:** Is there a sequence $(S_1, \ldots, S_\tau)$ such that for all $i \in \{1, \ldots, \tau\}$, $S_i$ is a size-at-most-$k$ vertex cover of $(V, E_i)$, and for all $i \in \{1, \ldots, \tau - 1\}$, $|S_i \triangle S_{i+1}| \leq \ell$?




$k = 2, \ \ell = 1$

# Multistage Vertex Cover: Results

| | general layers $0 \leq \ell < 2k$ | $\ell \geq 2k$ | tree layers $0 \leq \ell < 2k$ | one-edge layers $1 \leq \ell < 2$ |
|---|---|---|---|---|
| | NP-hard | NP-hard | NP-hard | NP-hard |
| $\tau$ | para-NP-hard | para-NP-hard | para-NP-hard | FPT, PK |
| $k$ | XP, W[1]-hard | FPT, No PK | XP, W[1]-hard | *open*, No PK |
| $k+\tau$ | FPT, PK | FPT, PK | FPT, PK | FPT, PK |

$\tau$: number of stages; $k$: allowed vertex cover size; $\ell$: allowed sym. diff. size

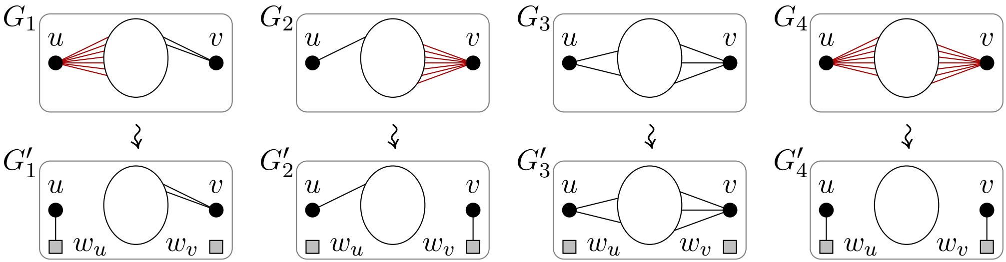
FPT: $f(p) \cdot |I|^{\mathcal{O}(1)}$-time;
XP: $|I|^{f(p)}$-time;
PK : $(I,p) \xrightarrow{\text{poly-time}} (I',p')$ with $|I'| + p' \leq p^{\mathcal{O}(1)}$;

W[1]-hard: presumably not FPT;
para-NP-hard: presumably not XP;
No PK: presumably no PK.

# An $\mathcal{O}(\tau \cdot k^2)$-sized Kernel for MSVC—Lifting the Classic

**Reduction Rule** (Isolated vertices): If $\exists\, v \in V$ such that $e \cap v = \emptyset \;\forall\, e \in E(\mathcal{G}_\downarrow)$, then delete $v$.

**Reduction Rule** (High-degree): If $\exists v \in V$ with $J \coloneqq \{i \in \{1,\dots,\tau\} \mid \deg_{G_i}(v) > k\} \neq \emptyset$, then add vertex $w_v$ to $V$ and for each $i \in J$, remove all edges incident to $v$ in $G_i$ and add edge $\{v, w_v\}$.



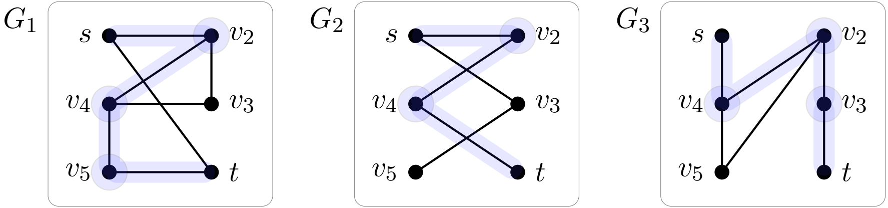

**Reduction Rule** (NO-instances): If above RRs are not applicable and $\exists$ layer with $> k^2$ edges, then output trivial NO-instance.

# Multistage $s$-$t$ Path

**Multistage $s$-$t$ Path (MSP)**

**Input:** A temporal graph $\mathcal{G} = (V, E_1, \ldots, E_\tau)$, two designated vertices $s, t \in V$, two integers $k, \ell \in \mathbb{N}$.

**Ques.:** Is there a sequence $(P_1, \ldots, P_\tau)$ such that for all $i \in \{1, \ldots, \tau\}$, $P_i$ is a order-at-most-$k$ $s$-$t$ path in $(V, E_i)$, and for all $i \in \{1, \ldots, \tau - 1\}$, $|V(P_i) \triangle V(P_{i+1})| \leq \ell$?

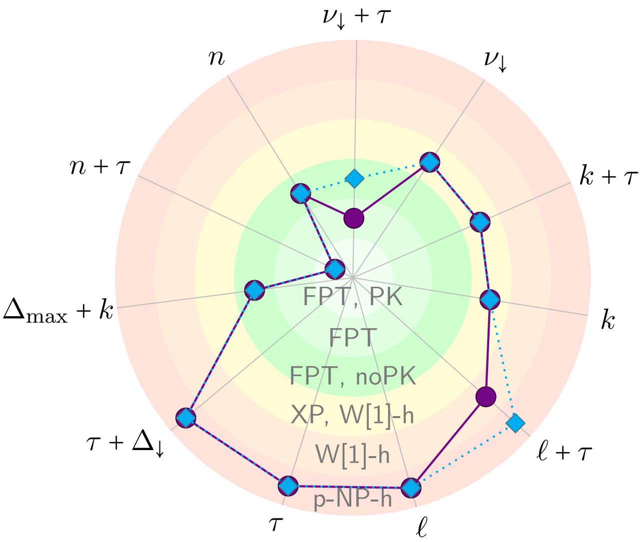


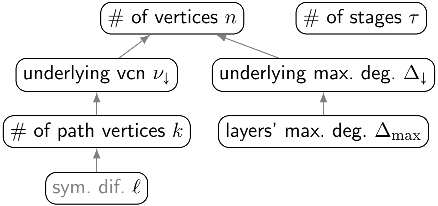

$k = 5,\ \ell = 1$

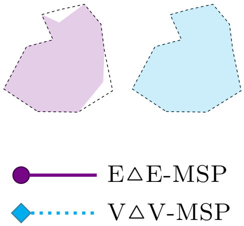
Application(s): Securing routes under uncertainty, robust re-routing, …

**Theorem:** NP-hard even for two stages and $\ell = 0$.

# Multistage $s$-$t$ Path: Results

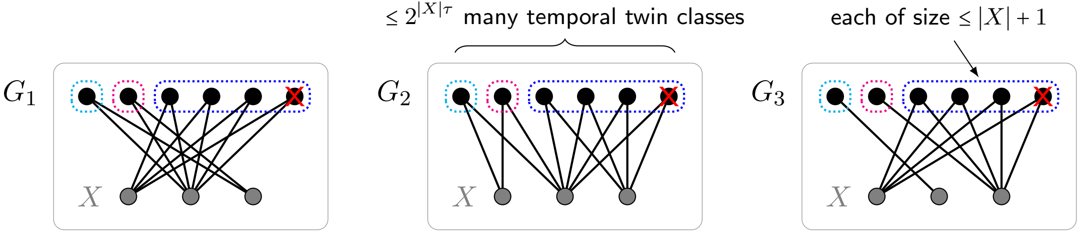


# A $O(4^{\nu_\downarrow \cdot \tau})$ Kernel for Multistage $s$-$t$ Path—Lifting Twins

**Definition.** Two vertices $v, w$ in a temporal graph $\mathcal{G}$ are called temporal twins if $N_{(V,E_i)}(v) = N_{(V,E_i)}(w)$ **for every** $i \in \{1, \ldots, \tau\}$.




**Kernelization:**

1. Compute vertex cover $X$ of $\mathcal{G}_\downarrow$ of size $\leq 2\nu_\downarrow$. (poly.-time)
2. Compute temporal twins in $V \setminus X$ of $\mathcal{G}$. (poly.-time)
3. Delete vertices in too large temporal twin classes. (poly.-time)

# Epilogue

Multistage is a generic and natural model.

Variations:

- Small over-all aggregated changes ("Global Multistage").
- Dissimilarity ($|\cdot \cap \cdot|$ small) or variety ($|\cdot \triangle \cdot|$ large).

Outlook:

- Between "standard" and "global": taking (time-)windows into account.
- Lifting more "classic" notions and techniques (e.g. for polynomial kernels for problem $L$ to MULTISTAGE $L$).

Three open problems restated in this talk:

- Is MULTISTAGE VERTEX COVER in FPT w.r.t. $k$ on temporal graphs with one-edge layers?
- Is $\mathrm{E}\triangle\mathrm{E}$-MSP in XP w.r.t. $\ell + \tau$?
- Does $\mathrm{E}\triangle\mathrm{E}$-MSP admit a poly. problem kernel w.r.t. $\nu_{\downarrow} + \tau$?

Thank you!

# References


[1] E. Bampis, B. Escoffier, and A. V. Kononov. LP-based algorithms for multistage minimization problems. *CoRR*, abs/1909.10354, 2019.

[2] E. Bampis, B. Escoffier, M. Lampis, and V. T. Paschos. Multistage matchings. In *Proceedings of 16th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT 2018)*, volume 101 of *LIPIcs*, pages 7:1–7:13. Schloss Dagstuhl—Leibniz-Zentrum für Informatik, 2018.

[3] E. Bampis, B. Escoffier, K. Schewior, and A. Teiller. Online multistage subset maximization problems. In M. A. Bender, O. Svensson, and G. Herman, editors, *27th Annual European Symposium on Algorithms, ESA 2019, September 9-11, 2019, Munich/Garching, Germany*, volume 144 of *LIPIcs*, pages 11:1–11:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.

[4] E. Bampis, B. Escoffier, and A. Teiller. Multistage knapsack. In *Proceedings of the 44th International Symposium on Mathematical Foundations of Computer Science (MFCS 2019)*, volume 138 of *LIPIcs*, pages 22:1–22:14. Schloss Dagstuhl—Leibniz-Zentrum für Informatik, 2019.

[5] R. Bredereck, T. Fluschnik, and A. Kaczmarczyk. Multistage committee election. *CoRR*, abs/2005.02300, 2020.

[6] M. Chimani, N. Troost, and T. Wiedera. Approximating multistage matching problems. *CoRR*, abs/2002.06887, 2020.

[7] D. Eisenstat, C. Mathieu, and N. Schabanel. Facility location in evolving metrics. In *Proceedings of 41st International Colloquium on Automata, Languages, and Programming (ICALP 2014)*, volume 8572 of *LNCS*, pages 459–470. Springer, 2014.

[8] T. Fluschnik, R. Niedermeier, V. Rohm, and P. Zschoche. Multistage vertex cover. In *Proceedings of the 14th International Symposium on Parameterized and Exact Computation (IPEC 2019)*, volume 148 of *LIPIcs*, pages 14:1–14:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.

[9] T. Fluschnik, R. Niedermeier, C. Schubert, and P. Zschoche. Multistage $s$-$t$ path: Confronting similarity with dissimilarity. *CoRR*, abs/2002.07569, 2020.

[10] A. Gupta, K. Talwar, and U. Wieder. Changing bases: Multistage optimization for matroids and matchings. In *Proceedings of 41st International Colloquium on Automata, Languages, and Programming (ICALP 2014)*, volume 8572 of *LNCS*, pages 563–575. Springer, 2014.

[11] K. Heeger, A. Himmel, F. Kammer, R. Niedermeier, M. Renken, and A. Sajenko. Multistage problems on a global budget. *CoRR*, abs/1912.04392, 2019.

# Multistage Vertex Cover is W[1]-hard w.r.t. $k$

Clique $\leq_{\text{fpt}}$ Multistage Vertex Cover with $\ell = 2$: