On Separators in Temporal Graphs

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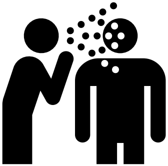
Algorithmic Aspects of Temporal Graphs,
Satellite Workshop of ICALP 2018, Prague

Based on joint work with Till Fluschnik, Rolf Niedermeier and Philipp Zschoche.

Introduction
Motivation: Separators



- Disease Spreading

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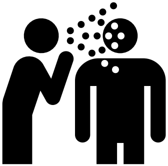

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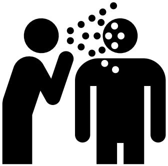

- Disease Spreading
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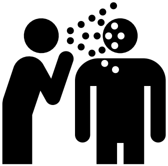
- Physical Proximity Networks



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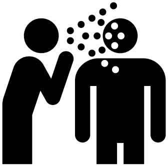

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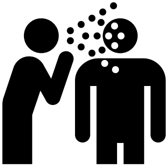

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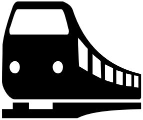

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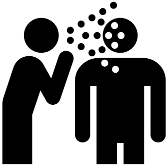

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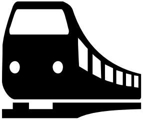

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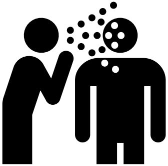

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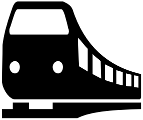

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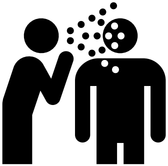

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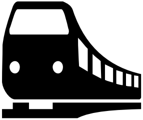

- Malware Spreading
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Temporal Graph

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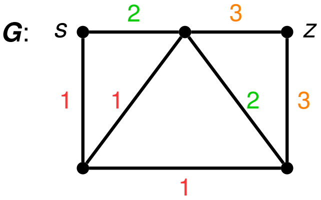

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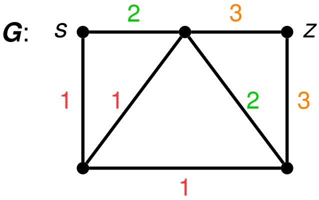

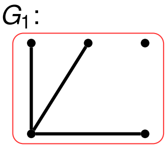

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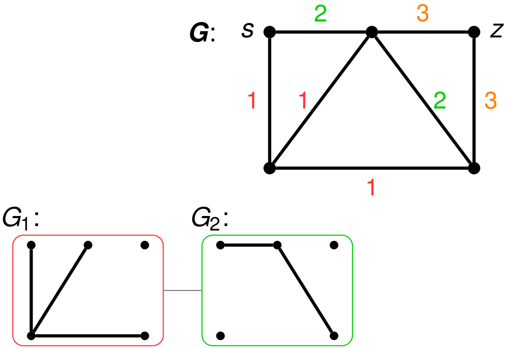

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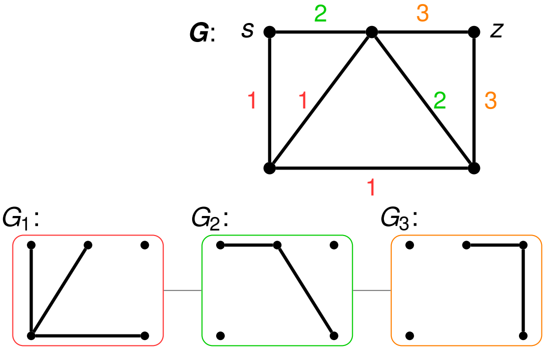

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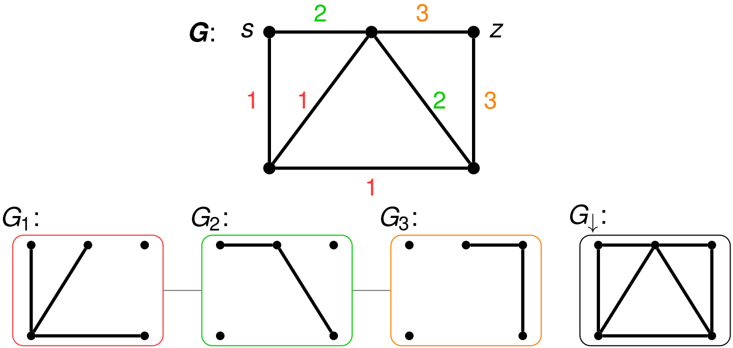

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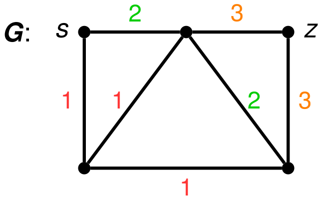

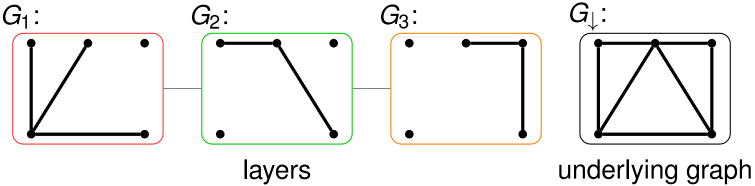

Temporal Paths

A **strict** (s, z)**-path** of length ℓ in $\boldsymbol{G} = (V, E_1, \ldots, E_\tau)$ is a list

$$P = ((\{s = v_0, v_1\}, t_1), \ldots, (\{v_{\ell-1}, v_\ell = z\}, t_\ell)),$$

where $\{v_{i-1}, v_i\} \in E_{t_i}$ for all $i \in [\ell]$ and $v_i \neq v_j$ for all $i, j \in \{0, \ldots, \ell\}$ with $i \neq j$ and for all $i \in [\ell - 1]$: $t_i < t_{i+1}$.

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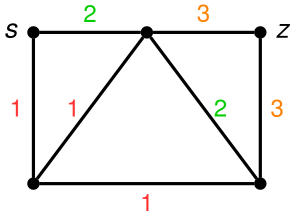

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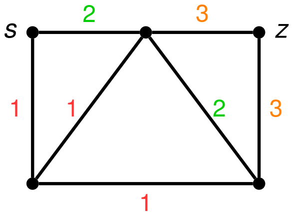

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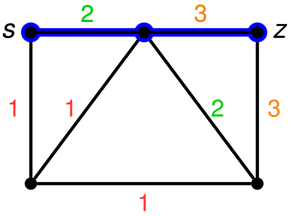

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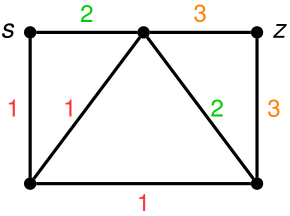

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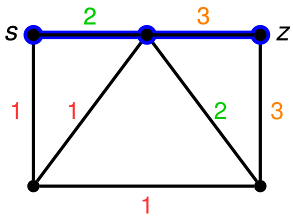

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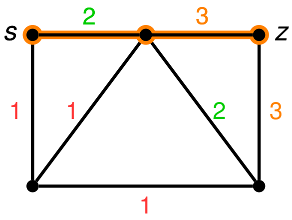

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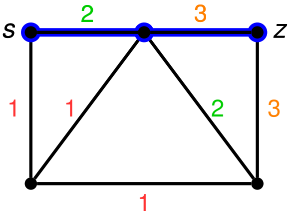

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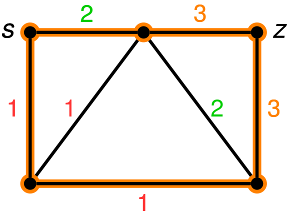

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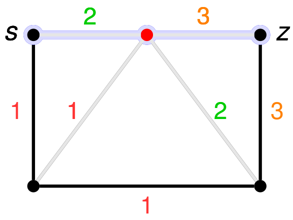

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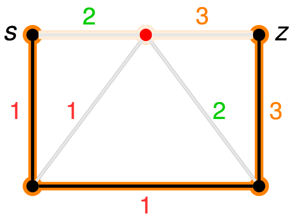

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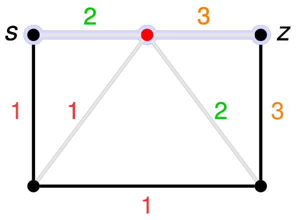

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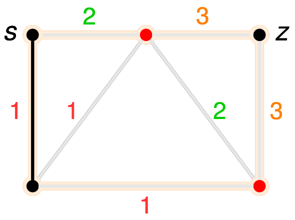

Strict (s, z)-Separation

Input: A temporal graph $\boldsymbol{G} = (V, E_1, \ldots, E_\tau)$ with two distinct vertices $s, z \in V$, and an integer k.
Question: Is there a subset $S \subseteq V \setminus \{s, z\}$ of size at most k such that there is no strict (s, z)-path in $\boldsymbol{G} - S$?

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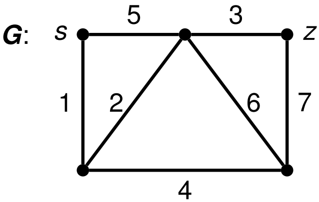

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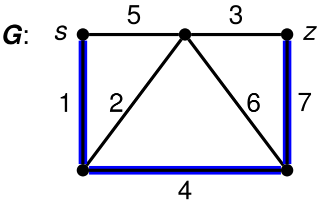

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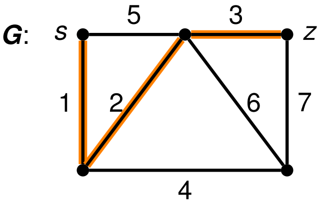

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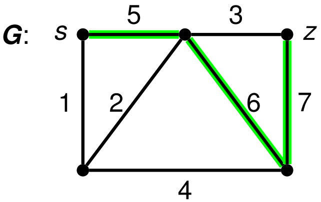

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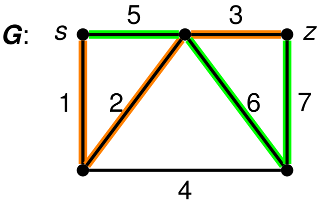

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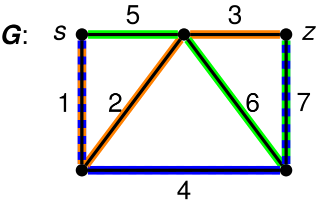

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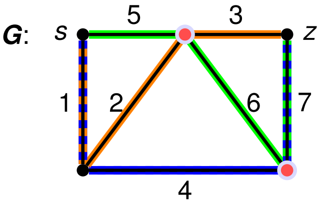

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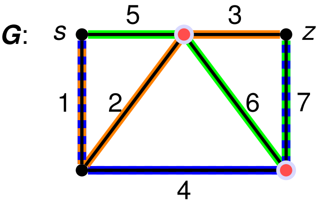
Question: Is there a subset $S \subseteq V \setminus \{s, z\}$ of size at most k such that there is no (non-)strict (s, z)-path in $\boldsymbol{G} - S$?

Berman [1996, Networks] showed that for temporal graphs Menger's Theorem **fails** (vertex-variant).



The edge-deletion variant can be computed in polynomial-time.

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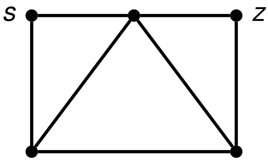

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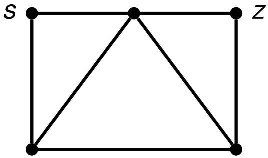

This presentation is based on Fluschnik et al. [2018, WG] and Zschoche et al. [2018, MFCS]. (Both to appear, available on arXiv.)

Parameterized Tractability

- **FPT** (fixed-parameter tractable): Solvable in $f(k) \cdot n^{O(1)}$ time.

n: instance size

k: parameter

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- **W[1]-hard**: Presumably no FPT algorithm (XP algorithm possible).

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Parameterized Hardness

- **W[1]-hard**: Presumably no FPT algorithm (XP algorithm possible).
- **para-NP-hard**: NP-hard for constant k (no XP algorithm).

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$2 \leq \tau \leq 4$	poly-time	para-NP-hard
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Complexity of Finding Temporal Separators

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Complexity of Finding Temporal Separators
Restricting each Layer

(Non-)Strict (s,z)-Separation with **restricted layers**.

Layer Restriction	Complexity

Complexity of Finding Temporal Separators
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Complexity of Finding Temporal Separators
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Take away message:
Layer restrictions do not help much.

Complexity of Finding Temporal Separators
Restricting the Underlying Graph

(Non-)Strict (s, z)-Separation with **restricted underlying graph.**

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Complexity of Finding Temporal Separators

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Take away message:

Underlying graph restrictions help sometimes.

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Complexity of Finding Temporal Separators
First Summary

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Complexity of Finding Temporal Separators

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All these restrictions are invariant under reordering of layers!

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Idea: Restrict "temporality" of the input graph.

Complexity of Finding Temporal Separators
Temporal Restrictions

Temporal graph classes with temporal aspects:

Restriction	(s,z)-Separation Strict	Non-Strict
p-monotone		

Definition (cf. Khodaverdian et al. [2016]; Casteigts et al. [2012])

$\boldsymbol{G} = (V, E_1, \dots, E_\tau)$ is p-**monotone** if there are $1 = i_1 < \dots < i_{p+1} = \tau$ such that for all $\ell \in [p]$ it holds that $E_j \subseteq E_{j+1}$ or $E_j \supseteq E_{j+1}$ for all $i_\ell \leq j < i_{\ell+1}$.

Complexity of Finding Temporal Separators
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q-periodic		

Definition (cf. Liu and Wu [2009]; Casteigts et al. [2012]; Flocchini et al. [2013])

$\boldsymbol{G} = (V, E_1, \ldots, E_\tau)$ is ***q*-periodic** if $E_i = E_{i+q}$ for all $i \in [\tau - q]$. We call $r := \tau/q$ the number of periods.

Complexity of Finding Temporal Separators

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Complexity of Finding Temporal Separators
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- Order-Preserving Temporal Unit Interval Graphs.
- Temporal Graph with bounded-sized Temporal Core.

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Recall: $<_V$ is compatible with a unit interval graph $G = (V, E)$ if $\{x, y\} \in E$ with $x <_V y$ implies $\{v \in V \mid x \leq_V v \leq_V y\}$ is a clique.

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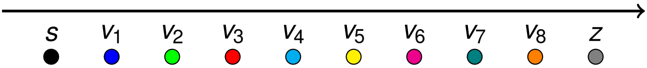
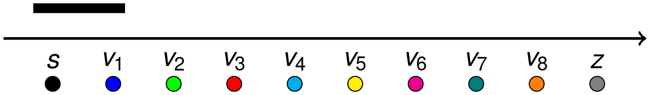
Motivation: Physical proximity networks in one-dimensional spaces.

(s, z)-Separation on Temporal Unit Interval Graphs

Poly-time Algo for Non-Strict (s, z)-Separation Order-Preserving Temporal Unit Interval Graphs



Vertex Ordering $<_V$



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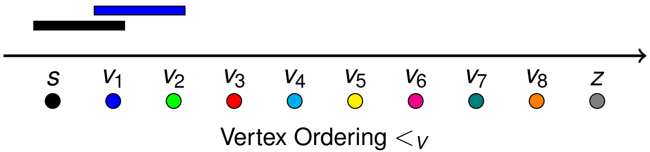

s
v1
v2
v3
v4
v5
v6
v7
v8
z
Vertex Ordering <V

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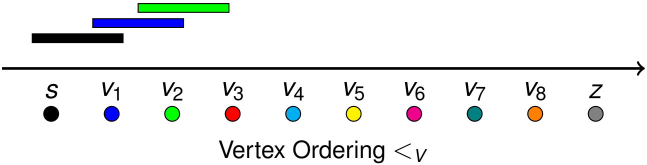

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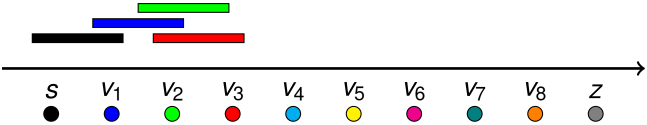

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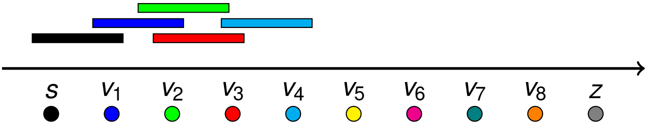

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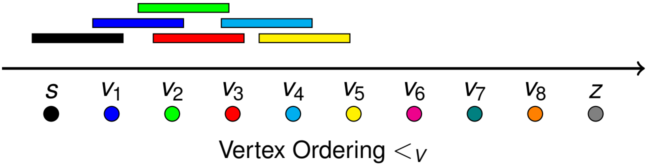

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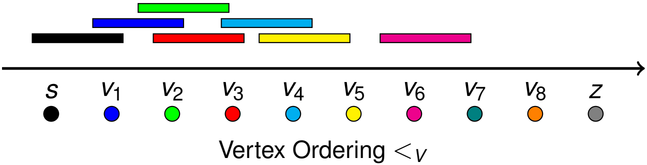

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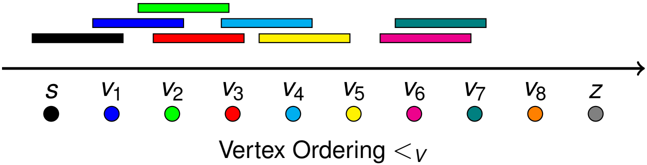

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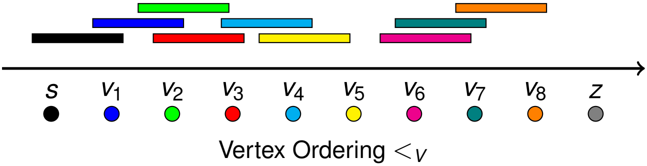

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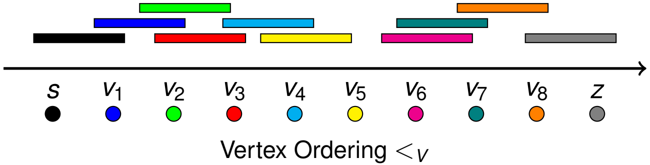

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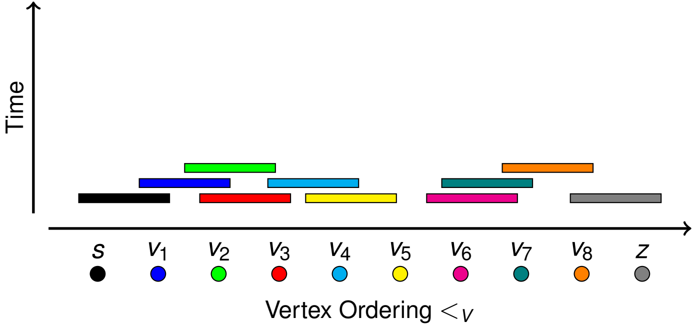

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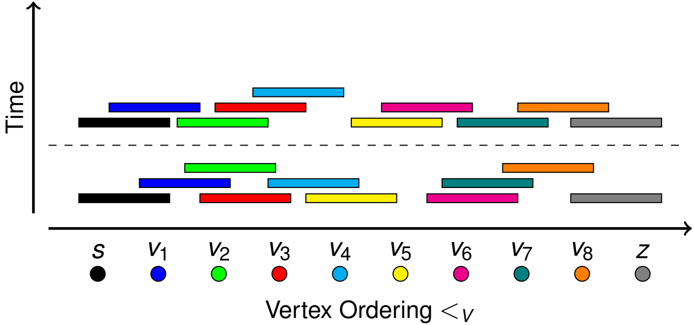
Poly-time Algo for Non-Strict (s, z)-Separation Order-Preserving Temporal Unit Interval Graphs



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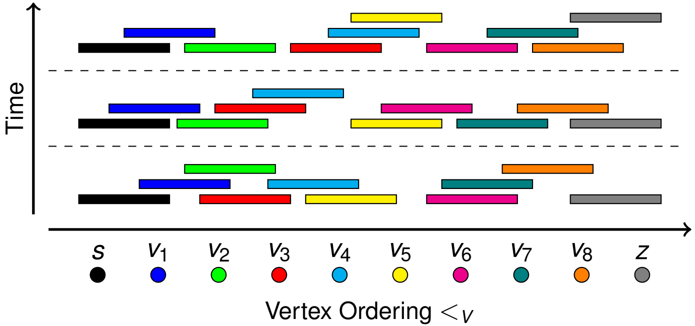

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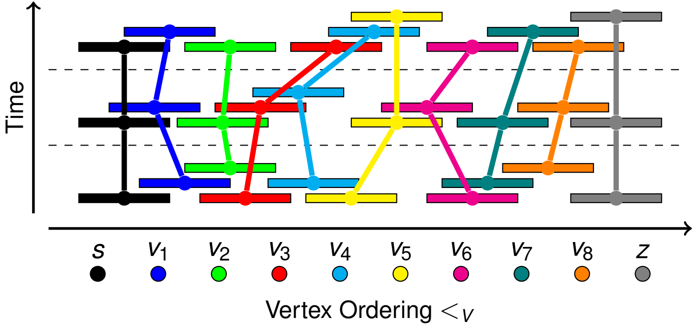

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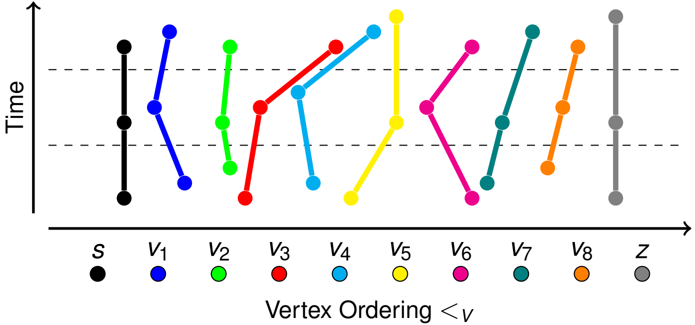

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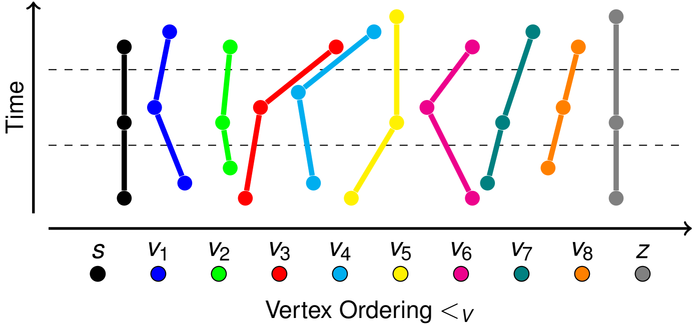

Observation

"Compatible" means these lines do not cross.

(s, z)-Separation on Temporal Unit Interval Graphs

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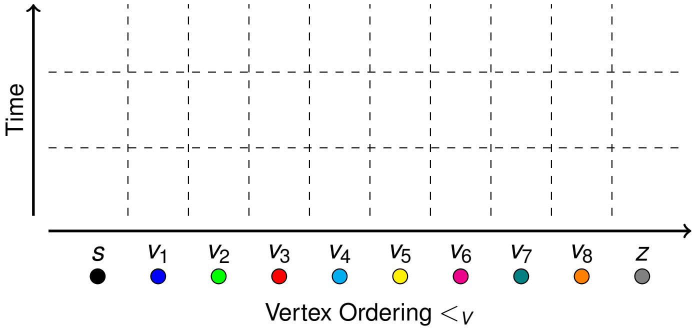

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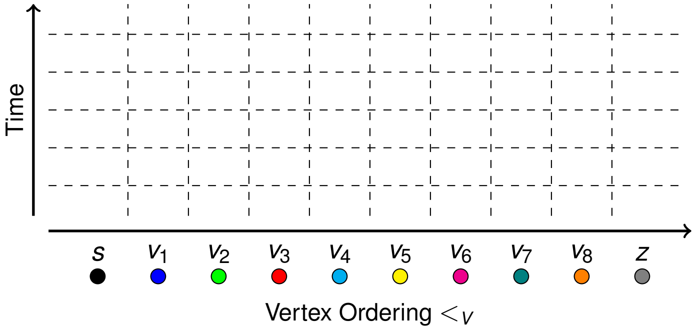

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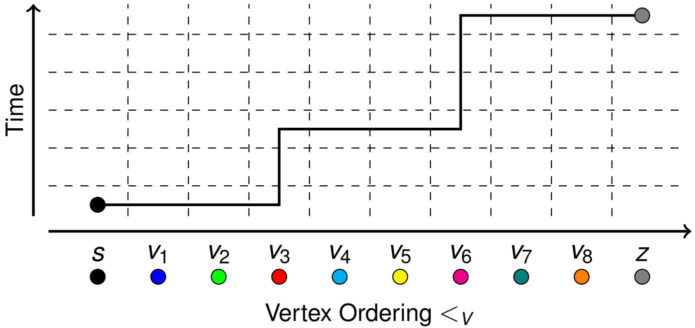
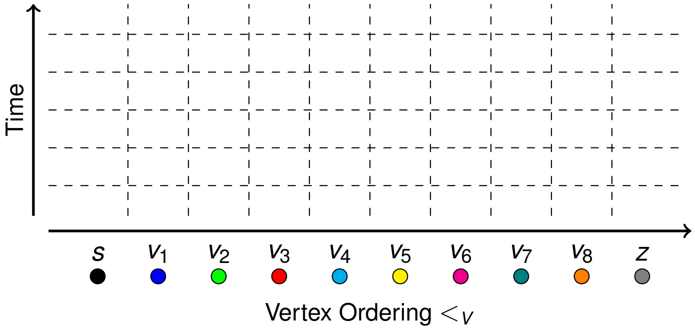


Observation

There are always temporal paths that follow the vertex ordering.

(s, z)-Separation on Temporal Unit Interval Graphs
Poly-time Algo for Non-Strict (s, z)-Separation Order-Preserving Temporal Unit Interval Graphs

DP-Table T:

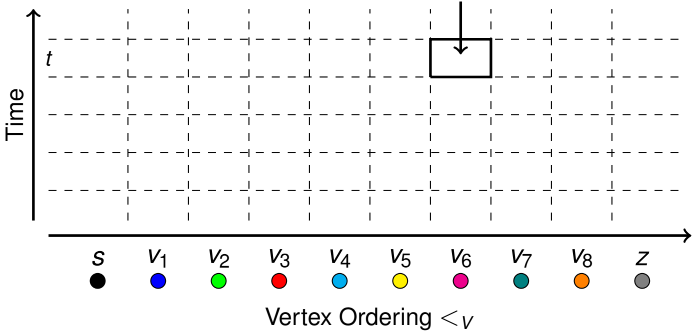


(s, z)-Separation on Temporal Unit Interval Graphs

Poly-time Algo for Non-Strict (s, z)-Separation Order-Preserving Temporal Unit Interval Graphs

DP-Table T:

$T[i, t] :=$ min. (s, z)-separator for time t, where no vertex "**behind**" v_i is reachable from s



(s,z)-Separation on Temporal Unit Interval Graphs

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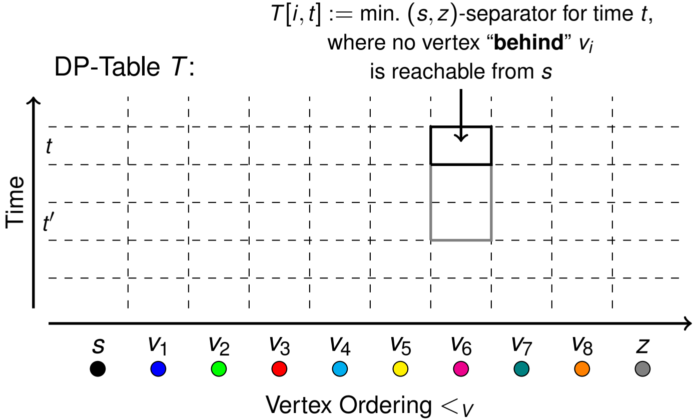
DP-Table T:



- Guess earliest time t' when v_i is reachable from s.

(s, z)-Separation on Temporal Unit Interval Graphs

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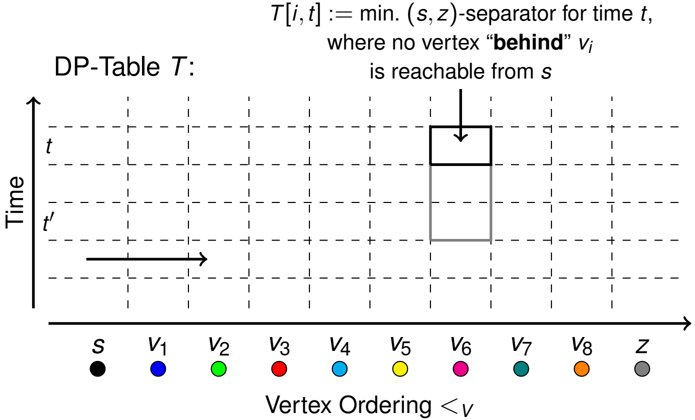
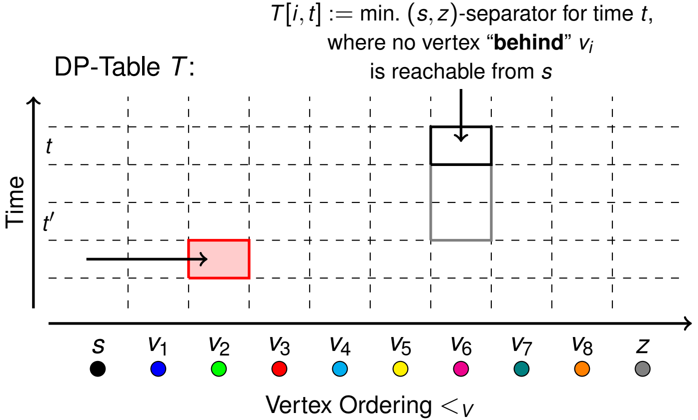


- Guess earliest time t' when v_i is reachable from s.
- Guess furthest vertex v_j reachable from s in $t' - 1$.

(s,z)-Separation on Temporal Unit Interval Graphs

Poly-time Algo for Non-Strict (s,z)-Separation Order-Preserving Temporal Unit Interval Graphs



- Guess earliest time t' when v_i is reachable from s.
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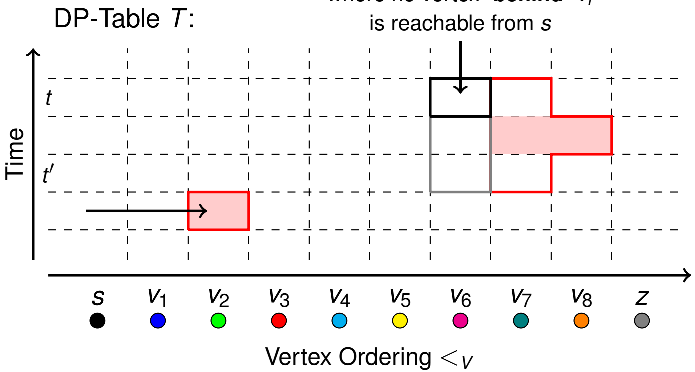
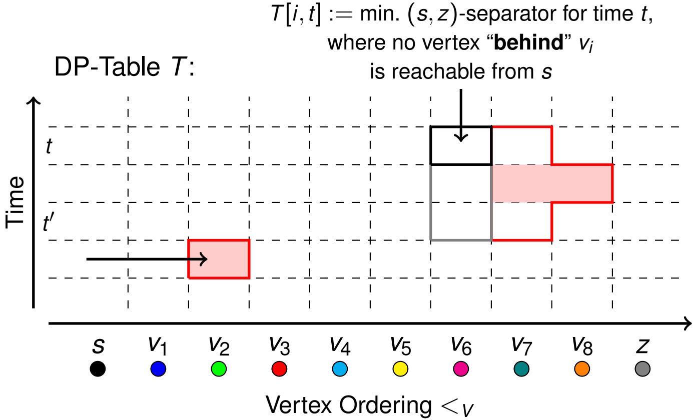


- Guess earliest time t' when v_i is reachable from s.
- Guess furthest vertex v_j reachable from s in $t' - 1$.
- $T[i, t] = T[j, t' - 1] \cup$ max. "right" neighborhood of v_i in $[t', t]$.

(s,z)-Separation on Temporal Unit Interval Graphs

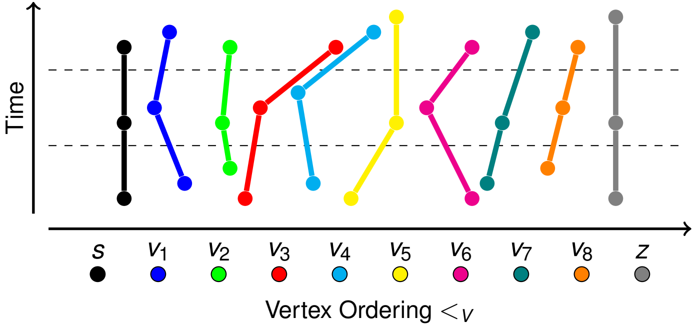
Poly-time Algo for Non-Strict (s,z)-Separation Order-Preserving Temporal Unit Interval Graphs



Theorem

Non-Strict (s,z)-Separation on order-preserving temporal unit interval graphs is poly-time solvable.

(s, z)-Separation on Temporal Unit Interval Graphs
Almost Order-Preserving Temporal Unit Interval Graphs



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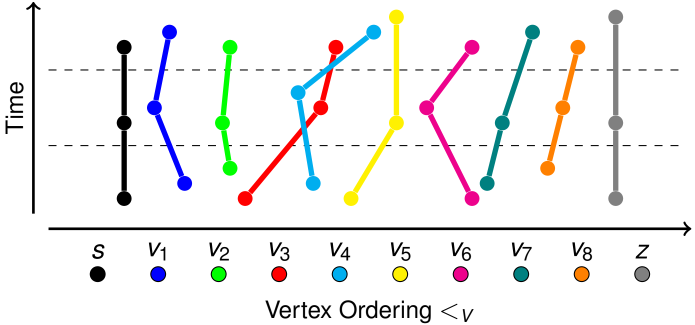

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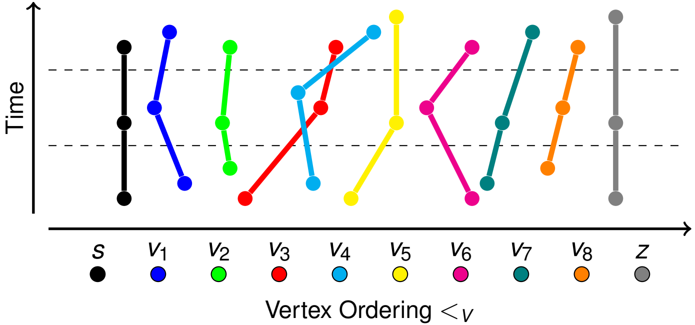

- Idea: Bound number of crossings between consecutive time steps.

  $\Leftrightarrow$ Vertex orderings have bounded **Kendall tau distance** κ.

(s, z)-Separation on Temporal Unit Interval Graphs

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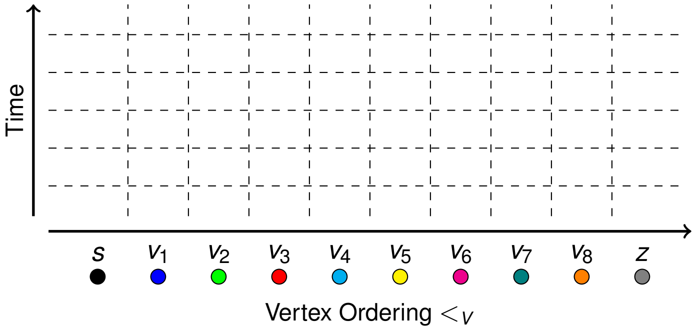

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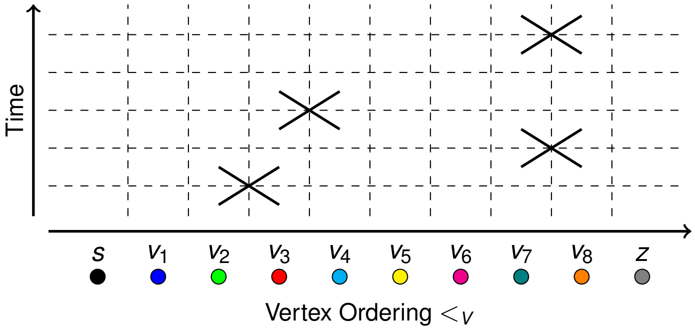

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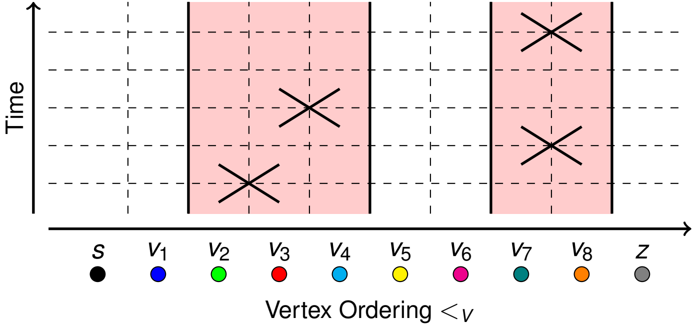

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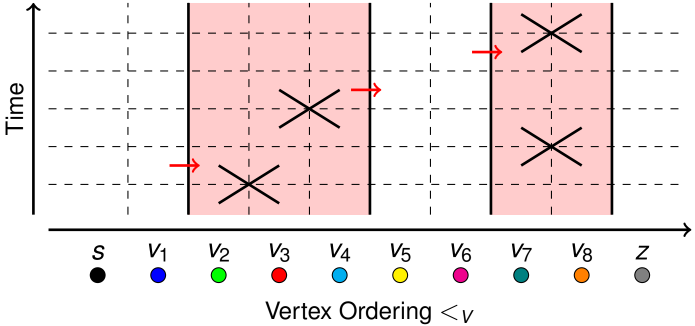

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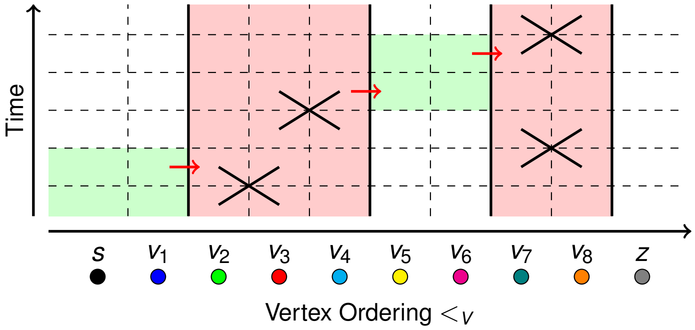

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(s, z)-Separation on Temporal Unit Interval Graphs

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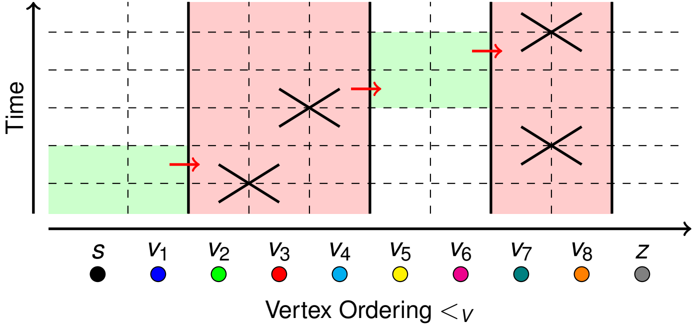

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- Size of regions bounded by κ and the lifetime τ.

(s, z)-Separation on Temporal Unit Interval Graphs
Summary

Theorem

(Non-)Strict (s, z)-Separation on order-preserving temporal unit interval graphs is poly-time solvable.

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Theorem

(Non-)Strict (s, z)-Separation on temporal unit interval graphs is FPT wrt. $(\kappa + \tau)$.

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Theorem

(Non-)Strict (s, z)-Separation on temporal unit interval graphs is para-NP-hard wrt. κ and para-NP-hard wrt. τ.

Temporal Core

The **temporal core** of $G = (V, E_1, \ldots, E_\tau)$ is the vertex set

$$W = \{v \in V \mid \exists \{v, w\} \in (\bigcup_{i=1}^{\tau} E_i) \setminus (\bigcap_{i=1}^{\tau} E_i)\}.$$

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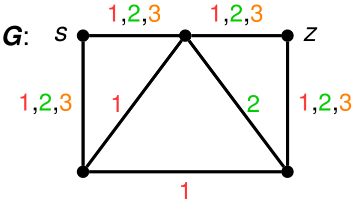

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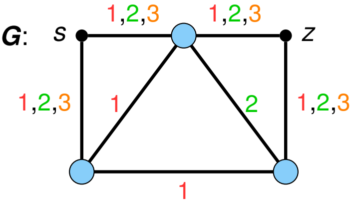

Temporal Core

Motivation and Definition

Temporal Core

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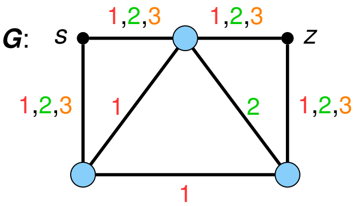
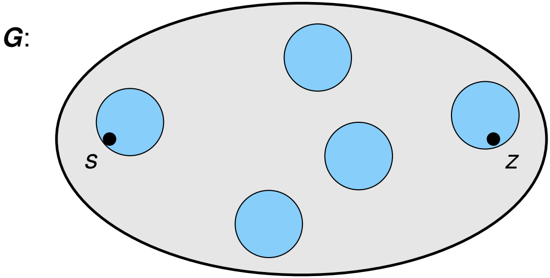


Recall: Strict (s, z)-Separation is NP-hard even if $W = \emptyset$.

Non-Strict (s, z)-Separation with small Temporal Cores

FPT Algorithm for "Size of the Temporal Core"

Given a temporal graph $\boldsymbol{G} = (V, E_1, \ldots, E_\tau)$ with temporal core W:



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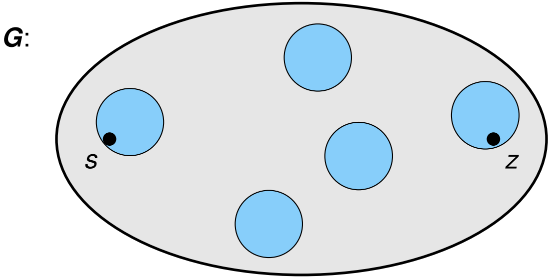
- Guess which core vertices are part of the separator.



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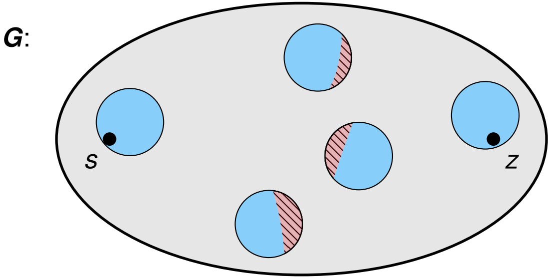

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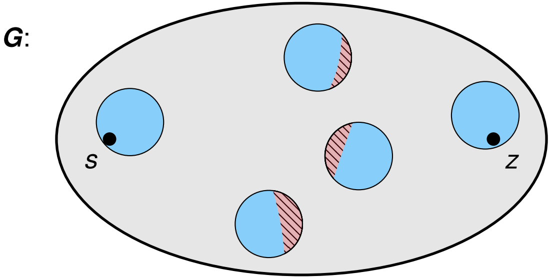

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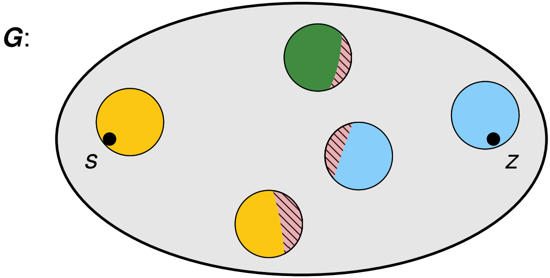

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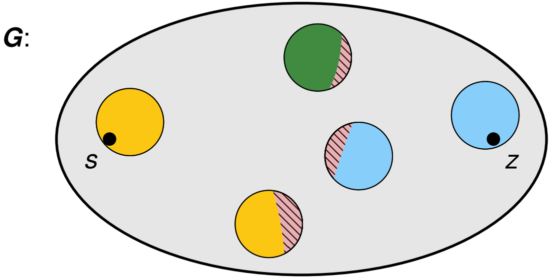
- Guess which core vertices are part of the separator.
- Guess which core vertices need to be separated from each other.
- Use an algorithm for **Node Multiway Cut**.



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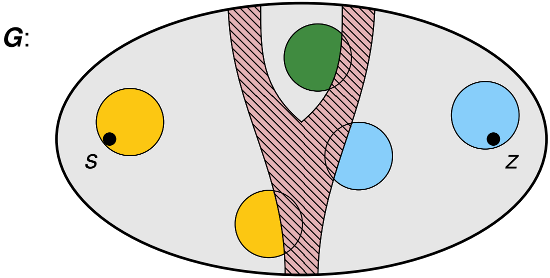

Node Multiway Cut

Input: An undirected graph $G = (V, E)$, a set of terminal $T \subseteq V$, and an integer k.
Question: Is there a set $S \subseteq (V \setminus T)$ of size at most k such there is no (t_1, t_2)-path for every distinct $t_1, t_2 \in T$?

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Theorem (Cygan et al. [2013], TOCT)

Node Multiway Cut can be solved in $2^{k-b} \cdot |V|^{O(1)}$ time,
where $b = \max_{x \in T} \min\{|S| \mid S \subseteq V \text{ is an } (x, T \setminus \{x\})\text{-separator}\}$.

Non-Strict (s, z)-Separation with small Temporal Cores

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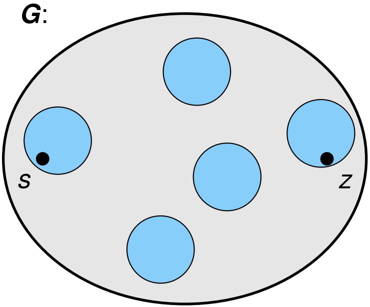
- Guess a set $S_W \subseteq (W \setminus \{s, z\})$ of size at most k.



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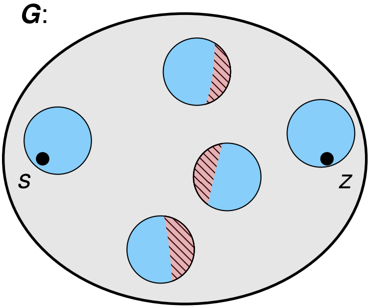

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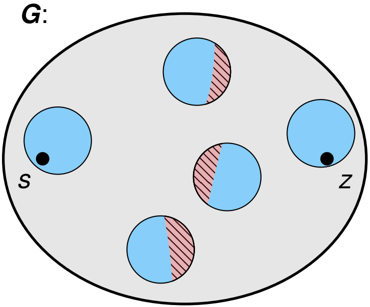

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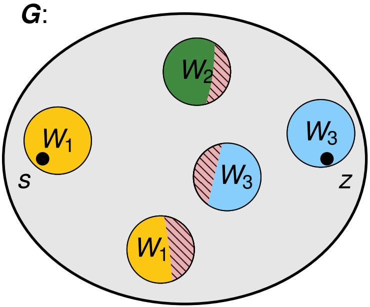

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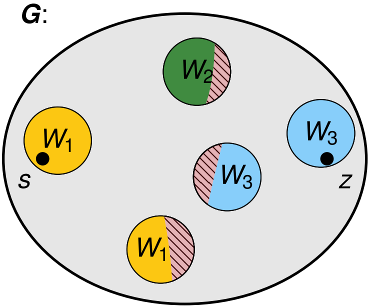

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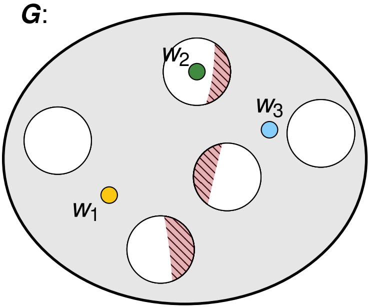

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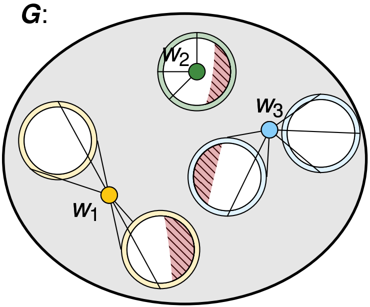
  For all $i \in [r]$, add edge sets $\{\{v, w_i\} \mid v \in N_{G_\downarrow}(W_i) \setminus W\}$.



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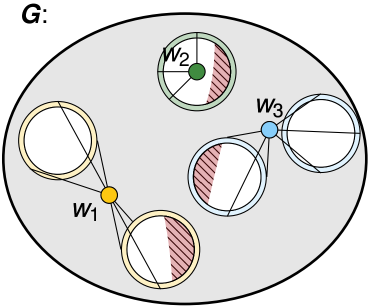

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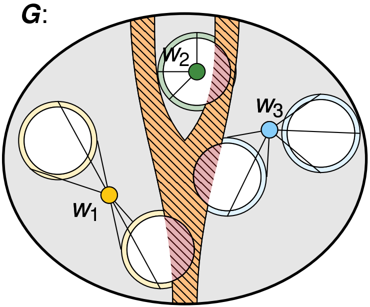

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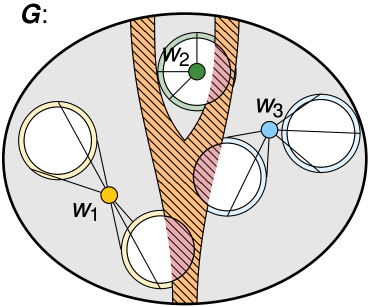

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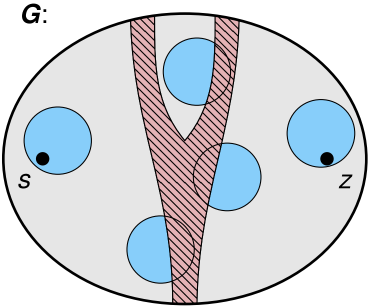

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Thank you!

References I

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Berman, K. A. (1996). Vulnerability of scheduled networks and a generalization of Menger's Theorem. *Networks*, 28(3):125–134.

Casteigts, A., Flocchini, P., Quattrociocchi, W., and Santoro, N. (2012). Time-varying graphs and dynamic networks. *International Journal of Parallel, Emergent and Distributed Systems*, 27(5):387–408.

Cygan, M., Pilipczuk, M., Pilipczuk, M., and Wojtaszczyk, J. O. (2013). On multiway cut parameterized above lower bounds. *ACM Transactions on Computation Theory*, 5(1):3:1–3:11.

Flocchini, P., Mans, B., and Santoro, N. (2013). On the exploration of time-varying networks. *Theoretical Computer Science*, 469:53–68.

Fluschnik, T., Molter, H., Niedermeier, R., and Zschoche, P. (2018). Temporal graph classes: A view through temporal separators. In *Proceedings of the 44th International Workshop on Graph-Theoretic Concepts in Computer Science (WG'18)*, LNCS. Springer. Accepted for publication. To appear.

Kempe, D., Kleinberg, J., and Kumar, A. (2002). Connectivity and inference problems for temporal networks. *Journal of Computer and System Sciences*, 64(4):820–842.

Khodaverdian, A., Weitz, B., Wu, J., and Yosef, N. (2016). Steiner network problems on temporal graphs. *CoRR*, abs/1609.04918v2.

Kuhn, F., Lynch, N. A., and Oshman, R. (2010). Distributed computation in dynamic networks. In *Proceedings of the 42nd Annual ACM Symposium on the Theory of Computing (STOC '10)*, pages 513–522. ACM.

Liu, C. and Wu, J. (2009). Scalable routing in cyclic mobile networks. *IEEE Transactions on Parallel and Distributed Systems*, 20(9):1325–1338.

Zschoche, P., Fluschnik, T., Molter, H., and Niedermeier, R. (2018). The complexity of finding small separators in temporal graphs. In *Proceedings of the 43rd International Symposium on Mathematical Foundations of Computer Science (MFCS'18)*, LIPIcs. Schloss Dagstuhl—Leibniz Center for Informatics. Accepted for publication. To appear.