Exploration of temporal graphs with bounded degree

Thomas Erlebach and Jakob Spooner

University of Leicester
{te17|jts21}@leicester.ac.uk

ICALP Workshop
"Algorithmic Aspects of Temporal Graphs"
9 July 2018

Outline

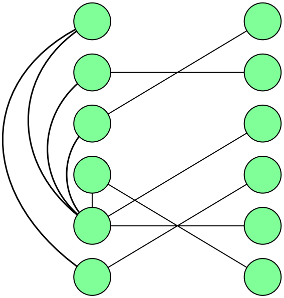
1. Temporal graphs
2. Temporal graph exploration problem (TEXP)
3. Known results
   - Instances that require $\Omega(n^2)$ steps
4. Faster exploration of degree-bounded graphs
5. Conclusions

Temporal Graphs

Temporal graph (Dynamic, time-varying graph)

A graph in which the edge set can change in every (time) step.

Step 0:

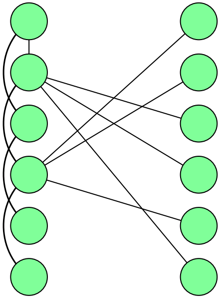


Temporal Graphs

Temporal graph (Dynamic, time-varying graph)

A graph in which the edge set can change in every (time) step.

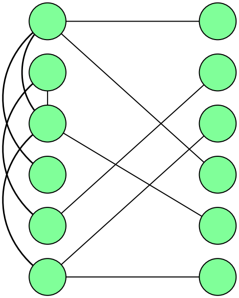
Step 1:



Temporal graph (Dynamic, time-varying graph)

A graph in which the edge set can change in every (time) step.

Step 2:



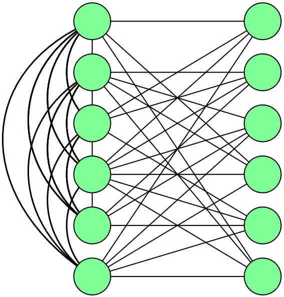
Temporal Graphs

Temporal graph (Dynamic, time-varying graph)

A graph in which the edge set can change in every (time) step.

Underlying graph

The graph with all edges that are present in at least one step.



Temporal (Time-Respecting) Path

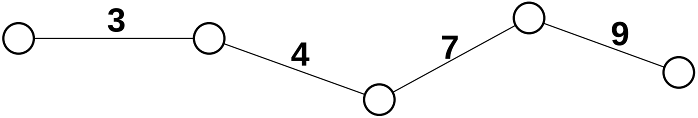
Time edge

A pair (e, t) where e is an edge of the underlying graph and t is a time step when e is present.

Temporal path (journey)

A sequence of time edges (e_1, t_1), ..., (e_k, t_k) such that $(e_1, e_2, \dots, e_k)$ is a path in the underlying graph and $t_1 < t_2 < \cdots < t_k$.

Example:



Temporal (Time-Respecting) Path

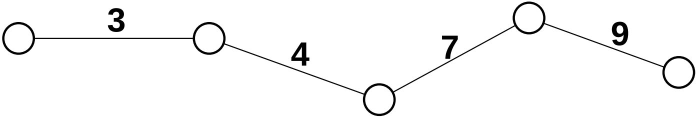
Time edge

A pair (e, t) where e is an edge of the underlying graph and t is a time step when e is present.

Temporal path (journey)

A sequence of time edges $(e_1, t_1), \ldots, (e_k, t_k)$ such that $(e_1, e_2, \ldots, e_k)$ is a path in the underlying graph and $t_1 < t_2 < \cdots < t_k$.

Example:



Temporal walk: temporal path where vertices may repeat

Temporal Graph Exploration

Temporal graph exploration problem (TEXP)

Starting at a given vertex s at time 0, find a fastest temporal walk that visits all vertices.
Equivalently: Schedule an agent: In each time step, traverse an edge or wait. Minimize time when last vertex is visited.

Temporal Graph Exploration

Temporal graph exploration problem (TEXP)

Starting at a given vertex s at time 0, find a fastest temporal walk that visits all vertices.
Equivalently: Schedule an agent: In each time step, traverse an edge or wait. Minimize time when last vertex is visited.

We assume: The whole temporal graph is known in advance.

Temporal Graph Exploration

Temporal graph exploration problem (TEXP)

Starting at a given vertex s at time 0, find a fastest temporal walk that visits all vertices.
Equivalently: Schedule an agent: In each time step, traverse an edge or wait. Minimize time when last vertex is visited.

We assume: The whole temporal graph is known in advance.

Michail and Spirakis [MFCS'14]

It is NP-complete to decide if a temporal graph can be explored if it need not be connected in each step.

Temporal Graph Exploration

Temporal graph exploration problem (TEXP)

Starting at a given vertex s at time 0, find a fastest temporal walk that visits all vertices.
Equivalently: Schedule an agent: In each time step, traverse an edge or wait. Minimize time when last vertex is visited.

We assume: The whole temporal graph is known in advance.

Michail and Spirakis [MFCS'14]

It is NP-complete to decide if a temporal graph can be explored if it need not be connected in each step.

$\Rightarrow$ Like Michail and Spirakis, we consider temporal graphs that are connected in each step and have lifetime $\geq n^2$.
(Note: We consider undirected graphs only.)

Temporal Graph Exploration

Temporal graph exploration problem (TEXP)

Starting at a given vertex s at time 0, find a fastest temporal walk that visits all vertices.
Equivalently: Schedule an agent: In each time step, traverse an edge or wait. Minimize time when last vertex is visited.

We assume: The whole temporal graph is known in advance.

Reachability lemma: Let $\mathcal{G}$ be a temporal graph with n vertices.

Agent can reach any vertex v from vertex u in n time steps.

Proof. Since $\mathcal{G}$ always has a u-v path, the set of vertices reachable from u increases in each step until v is reached.

Temporal Graph Exploration

Temporal graph exploration problem (TEXP)

Starting at a given vertex s at time 0, find a fastest temporal walk that visits all vertices.
Equivalently: Schedule an agent: In each time step, traverse an edge or wait. Minimize time when last vertex is visited.

We assume: The whole temporal graph is known in advance.

Reachability lemma: Let $\mathcal{G}$ be a temporal graph with n vertices.

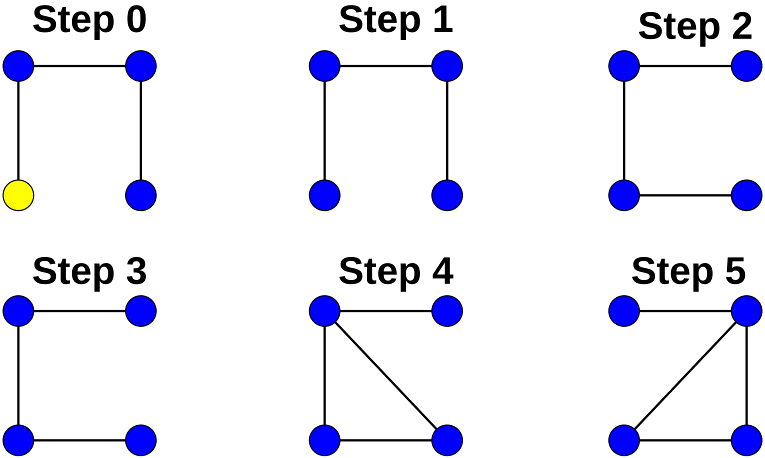
Agent can reach any vertex v from vertex u in n time steps.

Corollary

Any temporal graph can be explored in n^2 time steps.

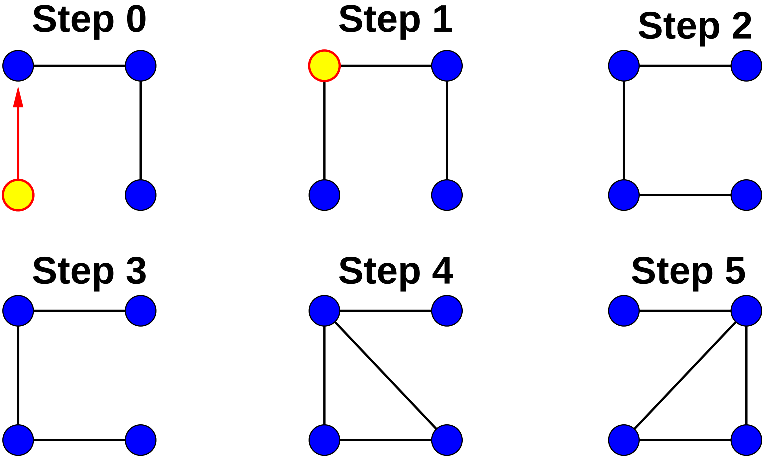
Example

Instance of Temporal Graph Exploration problem:



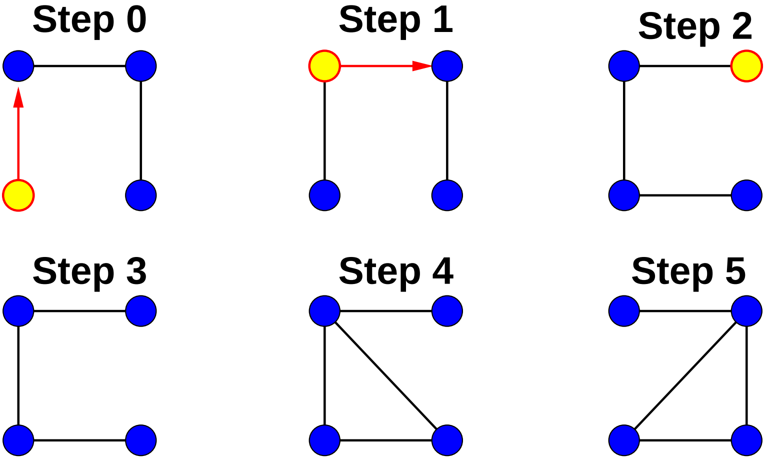
Example

Instance of Temporal Graph Exploration problem:



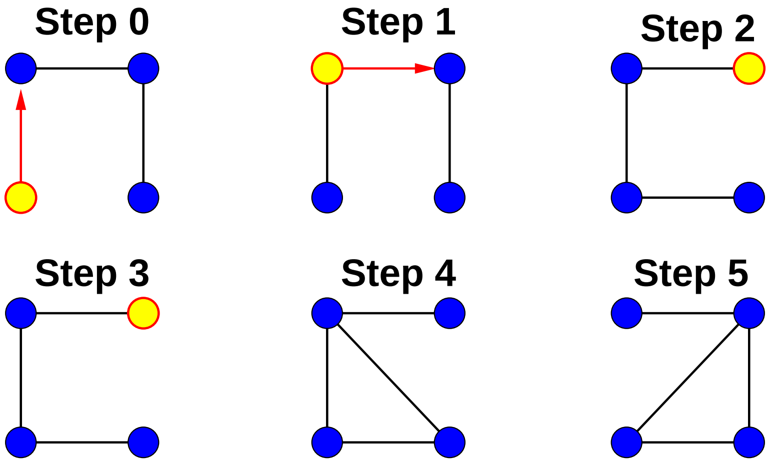
Example

Instance of Temporal Graph Exploration problem:



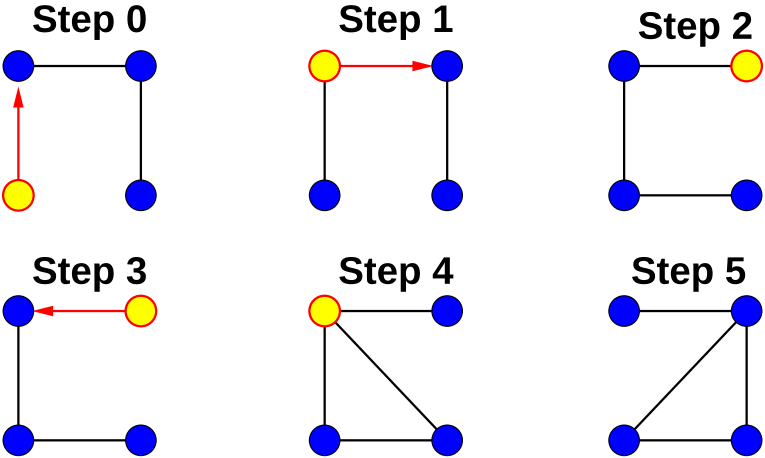
Example

Instance of Temporal Graph Exploration problem:



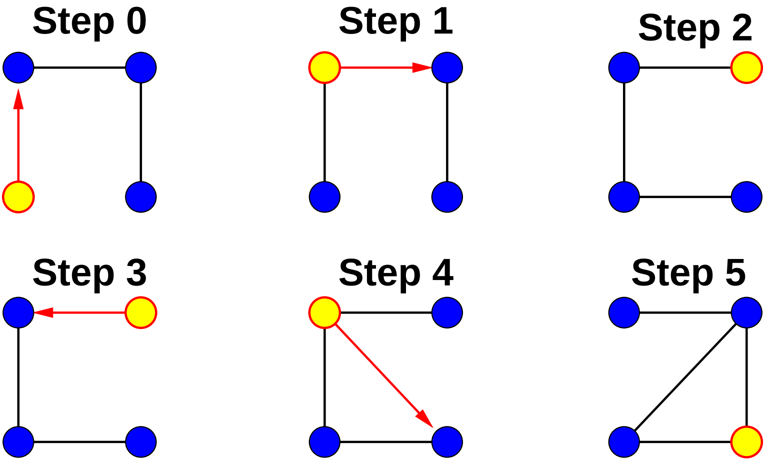
Example

Instance of Temporal Graph Exploration problem:



Example

Temporal exploration completed in Step 5.



Previous Work on TEXP

Avin, Koucký, Lotker, ICALP'08:

- Analyze cover time of random walk in temporal graph (with self-loops)
- Star construction shows that simple random walk may take $\Omega(2^n)$ steps
- Lazy random walk that leaves v only with probability $\deg(v)/(\Delta+1)$ has cover time $O(\Delta^2 n^3 \log^2 n)$

Michail and Spirakis, MFCS'14:

- D-approximation algorithm for temporal graph exploration, where D is the dynamic diameter
  Note: $1 \le D \le n-1$, can be equal to $n-1$
- No $(2-\varepsilon)$-approximation algorithm unless $P = NP$
- $(1.7+\varepsilon)$-approximation algorithm for temporal TSP with dynamic edge weights in $\{1, 2\}$

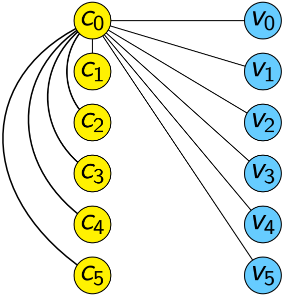
Previous Work on TEXP

E, Hoffmann, Kammer, ICALP'15:

- Instances of TEXP that require $\Omega(n^2)$ steps
- No $O(n^{1-\varepsilon})$-approximation algorithm unless $P = NP$
- Results for restricted underlying graphs:
  - treewidth k: $O(n^{1.5}k^{1.5}\log n)$ steps
  - planar: $O(n^{1.8}\log n)$ steps
  - cycle, cycle with chord: $O(n)$ steps
  - $2 \times n$ grid: $O(n\log^3 n)$ steps
  - Instances of TEXP where underlying graph is planar with $\Delta = 4$ that require $\Omega(n\log n)$ steps
- Further results on temporal graphs with randomly present edges or regularly present edges.

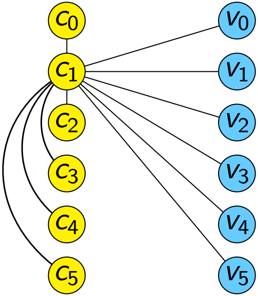
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step. Let c_0 be the center of a star in step 0.



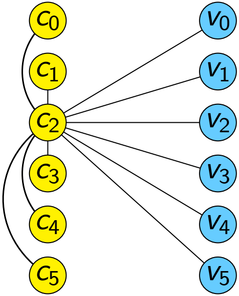
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step.
Let c_1 be the center of a star in step 1.



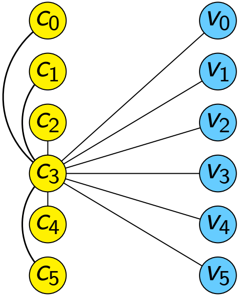
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step. Let c_2 be the center of a star in step 2.



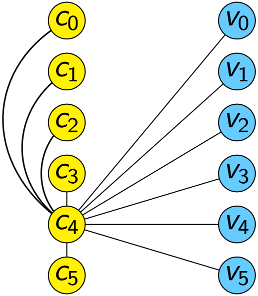
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step.
Let c_3 be the center of a star in step 3.



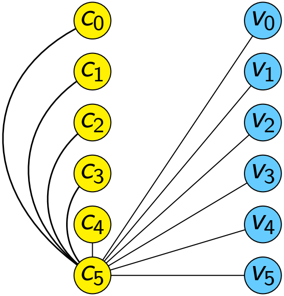
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step. Let c_4 be the center of a star in step 4.



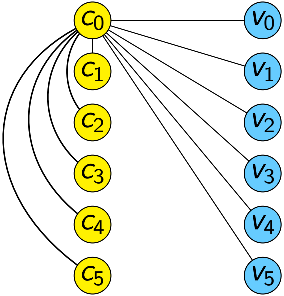
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step. Let c_5 be the center of a star in step 5.



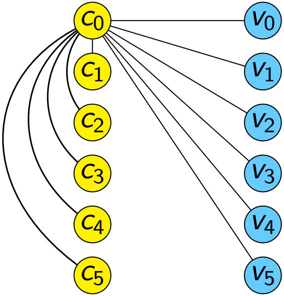
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step.
Let c_0 be the center of a star in step 6.



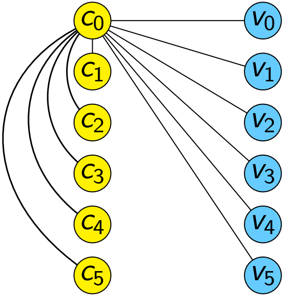
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step. Let c_i be the center of a star in step $i, \frac{n}{2}+i, n+i, \ldots$



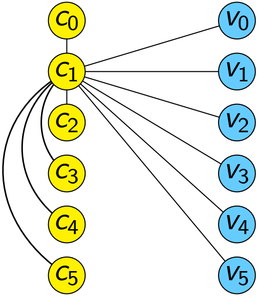
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step.
Let c_i be the center of a star in step $i, \frac{n}{2} + i, n + i, \ldots$
Agent starts in c_0. Let us only focus on exploring ●.



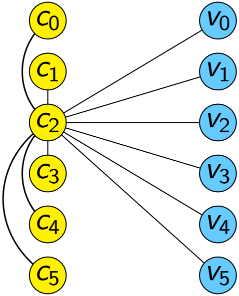
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step.
Let c_i be the center of a star in step $i, \frac{n}{2} + i, n + i, \ldots$
Agent starts in c_0. Let us only focus on exploring ⬤.



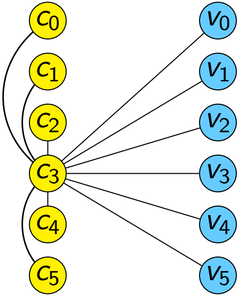
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step.
Let c_i be the center of a star in step $i, \frac{n}{2} + i, n + i, \ldots$
Agent starts in c_0. Let us only focus on exploring ⬤.



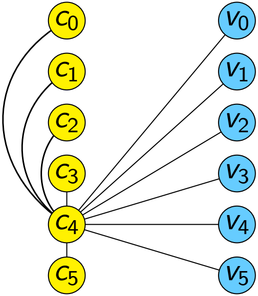
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step.
Let c_i be the center of a star in step $i, \frac{n}{2} + i, n + i, \ldots$
Agent starts in c_0. Let us only focus on exploring ●.



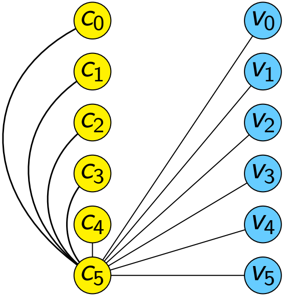
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step.
Let c_i be the center of a star in step $i, \frac{n}{2} + i, n + i, \ldots$
Agent starts in c_0. Let us only focus on exploring ⬤.



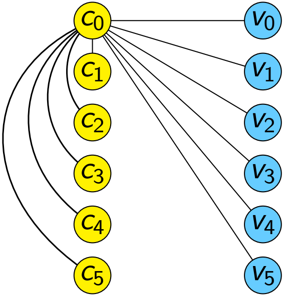
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step.
Let c_i be the center of a star in step $i, \frac{n}{2} + i, n + i, \ldots$
Agent starts in c_0. Let us only focus on exploring ●.



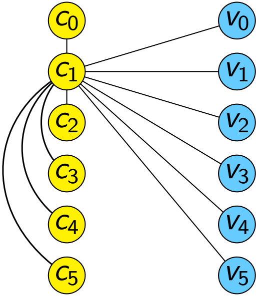
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step.
Let c_i be the center of a star in step $i, \frac{n}{2}+i, n+i, \ldots$
After returning to c_i, wait until c_i is center again.



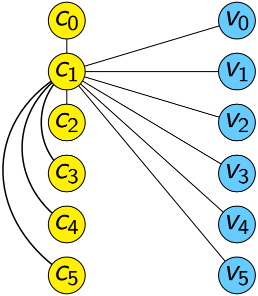
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step.
Let c_i be the center of a star in step $i, \frac{n}{2}+i, n+i, \ldots$
After returning to c_i, wait until c_i is center again.



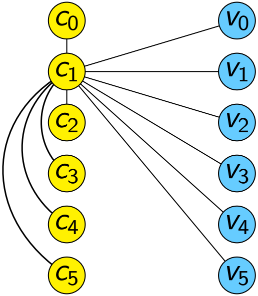
TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step.
Let c_i be the center of a star in step $i, \frac{n}{2}+i, n+i, \ldots$
After returning to c_i, wait until c_i is center again.
Each move from x to y with $x \neq y$: $\Omega(n)$ time steps.



TEXP instances that require $\Omega(n^2)$ steps

Consider the temporal graph below that is a star in each step.
Let c_i be the center of a star in step $i, \frac{n}{2} + i, n + i, \ldots$
After returning to c_i, wait until c_i is center again.
Each move from x to y with $x \neq y$: $\Omega(n)$ time steps.
In total, $\Omega(n^2)$ time steps.



Observations

The TEXP instances requiring $\Omega(n^2)$ steps have these properties:

- The underlying graph is very dense ($\Omega(n^2)$ edges).
- The graph in each step has a high-degree vertex (the center of the star has degree $n-1$).
- The graph changes in every step.

Questions

- What if we place a restriction on one of these?

Observations

The TEXP instances requiring $\Omega(n^2)$ steps have these properties:

- The underlying graph is very dense ($\Omega(n^2)$ edges).
- The graph in each step has a high-degree vertex (the center of the star has degree $n-1$).
- The graph changes in every step.

Questions

- What if we place a restriction on one of these?
- **Today:** What if the graph **in each step** has bounded degree?

Bounded Degree Graph Exploration

Temporal graph of bounded degree

A temporal graph $\mathcal{G}$ has degree bounded by Δ if the graph in each step has maximum degree at most Δ.

Question: What is the worst-case exploration time for temporal graphs of bounded degree?

We know:

- Upper bound $O(n^2)$ holds for arbitrary graphs.
- Lower bound $\Omega(n \log n)$ for underlying planar graphs with maximum degree $\Delta = 4$.

Main Result

Theorem

A temporal graph $\mathcal{G}$ with degree bounded by Δ can always be explored in

$$O\left(\log \Delta \cdot \frac{n^2}{\log n}\right)$$

steps.

Remarks:

- For $\log \Delta = o(\log n)$, the exploration time is $o(n^2)$.
- For $\Delta = O(1)$, the exploration time is $O(\frac{n^2}{\log n})$.
- There is still a huge gap to the lower bound of $\Omega(n \log n)$.

Proof Overview

Theorem

A temporal graph $\mathcal{G}$ with degree bounded by Δ can be explored in $O\left(\log \Delta \cdot \frac{n^2}{\log n}\right)$ steps.

Proof.

- While there are $\Omega(\frac{n}{\log_\Delta n})$ unexplored vertices, visit $O(\log_\Delta n)$ unexplored vertices in $O(n)$ steps.

$$\Rightarrow O\left(\frac{n}{\log_\Delta n} \cdot n\right) = O\left(\log \Delta \cdot \frac{n^2}{\log n}\right) \text{ steps}$$

- Visit the last $O(\frac{n}{\log_\Delta n})$ unexplored vertices in $O(n)$ steps per vertex.

$$\Rightarrow O\left(\frac{n}{\log_\Delta n} \cdot n\right) = O\left(\log \Delta \cdot \frac{n^2}{\log n}\right) \text{ steps}$$

Visiting many vertices quickly

Lemma (Main Lemma)

While there are $\Omega(\frac{n}{\log_\Delta n})$ unexplored vertices, we can visit $O(\log_\Delta n)$ unexplored vertices in $O(n)$ steps.

Proof idea.

- Assume current vertex is v, current step is t.
- Let U be the current set of unexplored vertices.
- **Claim:** There exists a walk W starting at some $u \in U$ at time $t + n$ that visits $O(\log_\Delta n)$ unexplored vertices in $O(n)$ steps.

$\Rightarrow$ Move from v to u during time t to $t + n$, then follow W.

Auxiliary Lemma

Lemma (Auxiliary Lemma)

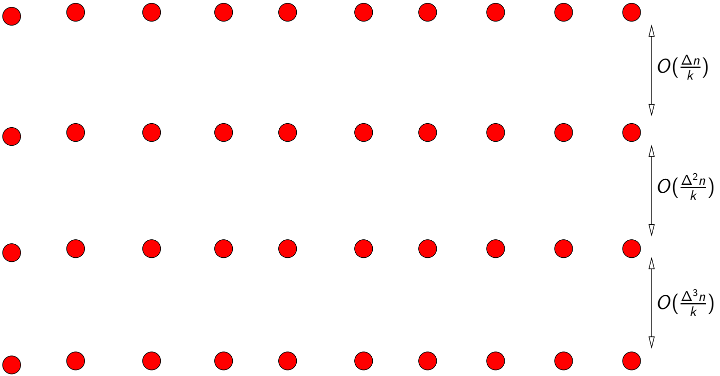
Let T be a set of $k = |T|$ unexplored vertices. There are $\Omega(\frac{k}{\Delta})$ disjoint pairs $(u, v) \in T^2$ s.t. u can reach v in $O(\frac{\Delta n}{k})$ steps.

Proof of Claim

Claim

There is a walk W starting at some $u \in U$ at time $t + n$ that visits $O(\log_\Delta n)$ unexplored vertices in $O(n)$ steps.

Proof sketch.

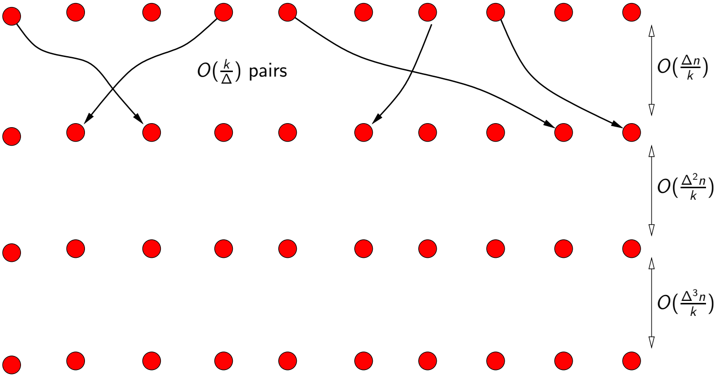


Proof of Claim

Claim

There is a walk W starting at some $u \in U$ at time $t + n$ that visits $O(\log_\Delta n)$ unexplored vertices in $O(n)$ steps.

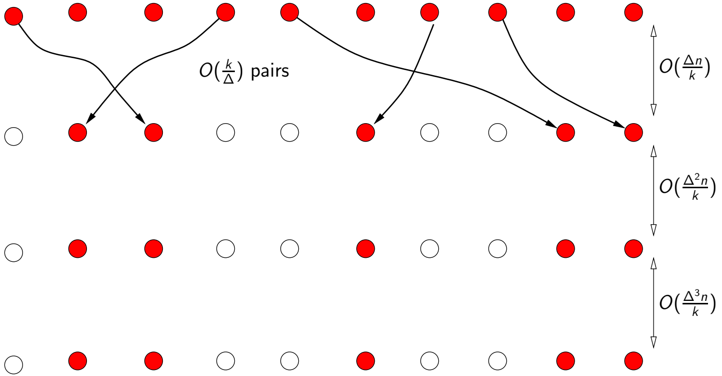
Proof sketch.



Claim

There is a walk W starting at some $u \in U$ at time $t + n$ that visits $O(\log_\Delta n)$ unexplored vertices in $O(n)$ steps.

Proof sketch.

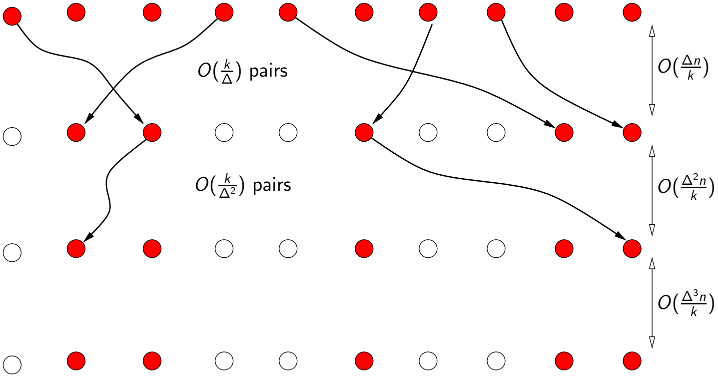


Proof of Claim

Claim

There is a walk W starting at some $u \in U$ at time $t + n$ that visits $O(\log_\Delta n)$ unexplored vertices in $O(n)$ steps.

Proof sketch.

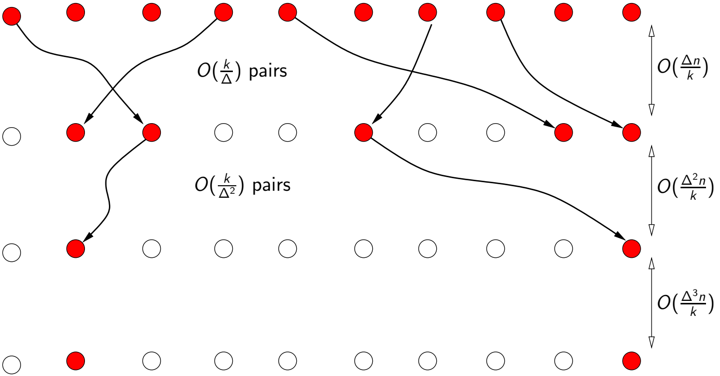


Proof of Claim

Claim

There is a walk W starting at some $u \in U$ at time $t+n$ that visits $O(\log_\Delta n)$ unexplored vertices in $O(n)$ steps.

Proof sketch.

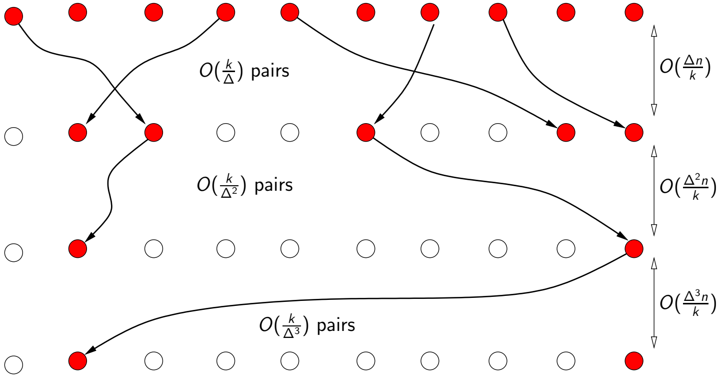


Proof of Claim

Claim

There is a walk W starting at some $u \in U$ at time $t + n$ that visits $O(\log_\Delta n)$ unexplored vertices in $O(n)$ steps.

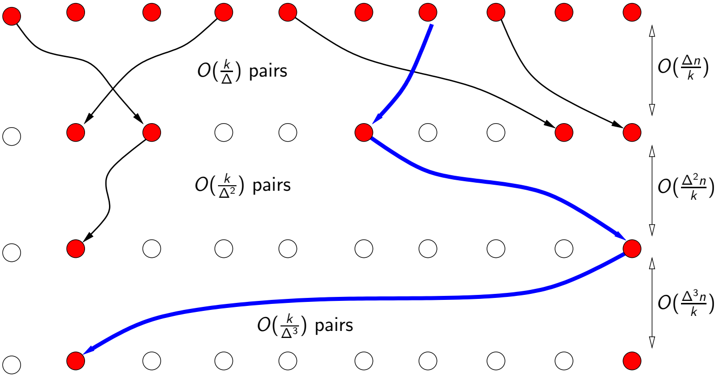
Proof sketch.



Claim

There is a walk W starting at some $u \in U$ at time $t+n$ that visits $O(\log_\Delta n)$ unexplored vertices in $O(n)$ steps.

Proof sketch.



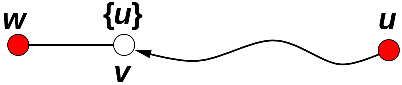
Proof of Auxiliary Lemma

Lemma (Auxiliary Lemma)

Let T be a set of $k = |T|$ unexplored vertices. There are $\Omega(\frac{k}{\Delta})$ disjoint pairs $(u, v) \in T^2$ s.t. u can reach v in $O(\frac{\Delta n}{k})$ steps.

Proof.

- Maintain a **home set** $H_v \subseteq T$ of each $v \in L = V \setminus T$:
  - $0 \leq |H_v| \leq 2$
  - Each $u \in H_v$ can reach v by the current time step.
- If a vertex $w \in T$ is adjacent to a vertex $v \in L$ with $u \in H_v$ for some $u \neq w$, a pair (u, w) is formed.



Potential function

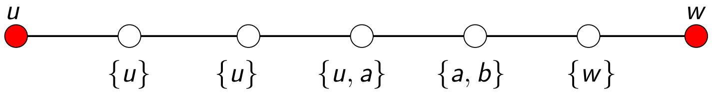
- Potential function $\Phi = \sum_{v \in L}(|H_v| + 1) \leq 3n$.
- We can show:
  - Φ increases by $\approx \frac{k}{2\Delta}$ in each step.
  - Formation of a pair decreases potential by at most $\frac{20\Delta n}{k}$.
- If fewer than $\frac{k}{20\Delta}$ pairs were formed in $\frac{10\Delta n}{k}$ steps, we would have

$$\Phi > \frac{10\Delta n}{k} \cdot \frac{k}{2\Delta} - \frac{k}{20\Delta} \cdot \frac{20\Delta n}{k} = 5n - n > 3n\,,$$

a contradiction.

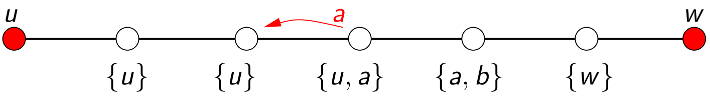
Obtaining the potential increase

- Consider spanning tree T of current graph.
- Find $\Omega(\frac{k}{\Delta})$ disjoint paths $P_{u,w}$ between vertices $u, w \in T$.
- On path $P_{u,w}$, increase potential of one vertex $v \in L$ by adding u or w to its home set H_v (and possibly adjusting other home sets).
- Example:



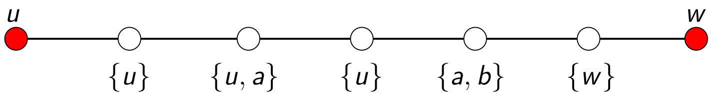
Obtaining the potential increase

- Consider spanning tree T of current graph.
- Find $\Omega(\frac{k}{\Delta})$ disjoint paths $P_{u,w}$ between vertices $u, w \in T$.
- On path $P_{u,w}$, increase potential of one vertex $v \in L$ by adding u or w to its home set H_v (and possibly adjusting other home sets).
- Example:



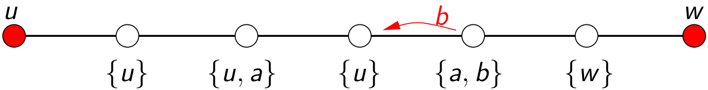
Obtaining the potential increase

- Consider spanning tree T of current graph.
- Find $\Omega(\frac{k}{\Delta})$ disjoint paths $P_{u,w}$ between vertices $u, w \in T$.
- On path $P_{u,w}$, increase potential of one vertex $v \in L$ by adding u or w to its home set H_v (and possibly adjusting other home sets).
- Example:



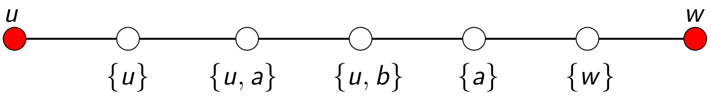
Obtaining the potential increase

- Consider spanning tree T of current graph.
- Find $\Omega(\frac{k}{\Delta})$ disjoint paths $P_{u,w}$ between vertices $u, w \in T$.
- On path $P_{u,w}$, increase potential of one vertex $v \in L$ by adding u or w to its home set H_v (and possibly adjusting other home sets).
- Example:



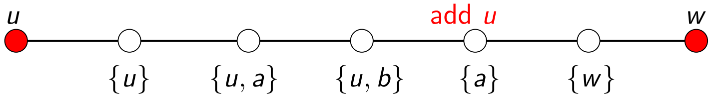
Obtaining the potential increase

- Consider spanning tree T of current graph.
- Find $\Omega(\frac{k}{\Delta})$ disjoint paths $P_{u,w}$ between vertices $u, w \in T$.
- On path $P_{u,w}$, increase potential of one vertex $v \in L$ by adding u or w to its home set H_v (and possibly adjusting other home sets).
- Example:



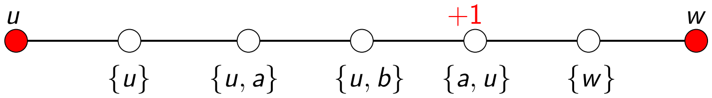
Obtaining the potential increase

- Consider spanning tree T of current graph.
- Find $\Omega(\frac{k}{\Delta})$ disjoint paths $P_{u,w}$ between vertices $u, w \in T$.
- On path $P_{u,w}$, increase potential of one vertex $v \in L$ by adding u or w to its home set H_v (and possibly adjusting other home sets).
- Example:



Obtaining the potential increase

- Consider spanning tree T of current graph.
- Find $\Omega(\frac{k}{\Delta})$ disjoint paths $P_{u,w}$ between vertices $u, w \in T$.
- On path $P_{u,w}$, increase potential of one vertex $v \in L$ by adding u or w to its home set H_v (and possibly adjusting other home sets).
- Example:



Conclusions

- We have shown that temporal graphs whose degree is bounded by Δ in each step can be explored in $O(\log \Delta \cdot \frac{n^2}{\log n})$ steps.
- The best known lower bound for small Δ is only $\Omega(n \log n)$ steps, so a large gap remains.
- We are still only at the beginning of understanding how restrictions on the underlying graph or on the graph in each step affect the worst-case exploration time.

Open Problems

- Close the gap for temporal graphs of bounded degree in each step.
- Exploration of temporal graphs whose underlying graph is planar:
  - What is the largest number of steps required?
    - Upper bound: $O(n^{1.8} \log n)$ steps
    - Lower bound: $\Omega(n \log n)$ steps
  - Approximation algorithms?
- Underlying graphs from other graph classes:
  - $n \times n$ grids
  - Planar graphs of bounded degree
  - Arbitrary graphs of bounded degree
- Instance-dependent lower bounds on exploration time
- Graphs that change only every $c > 1$ steps

Thank you!